

The Structure of Extreme Level Sets in Branching Brownian Motion

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Abstract

We study the structure of extreme level sets of a standard one dimensional branching Brownian motion, namely the sets of particles whose height is within a fixed distance from the order of the global maximum. It is well known that such particles congregate at large times in clusters of order-one genealogical diameter around local maxima which form a Cox process in the limit. We add to these results by finding the asymptotic size of extreme level sets and the typical height and shape of those clusters which carry such level sets. We also find the right tail decay of the distribution of the distance between the two highest particles. These results confirm two conjectures of Brunet and Derrida [13]. The proofs rely on studying the cluster distribution and should carry over to the branching random walk and the two-dimensional discrete Gaussian free field with no conceptual difficulty.

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1 Introduction and Results

1.1 Introduction

This work concerns the fine structure of extreme values of branching Brownian motion. The latter describes the motion of a particle which diffuses on the real line according to a standard Brownian motion for a time whose law is exponential with mean one and then splits into two independent child particles which repeat the same procedure starting from the last position of their parent.

One way of formulating this process is as follows. Take a continuous time (binary) Galton-Watson tree $T = (T_t : t \geq 0)$ with branching rate 1 and denote by L_t its set of leaves at time t , so that $\mathbb{E}|L_t| = e^t$. Then conditional on T let $h = (h_t(x) : t \geq 0, x \in L_t)$ be a mean-zero Gaussian process with covariance function given by

$$\mathbb{E}h_t(x)h_{t'}(x') = \sup\{s \geq 0 : x, x' \text{ share a common ancestor in } L_s\} : t, t' \geq 0, x \in L_t, x' \in L_{t'}.$$

The connection with the description above is then obtained by interpreting L_t as the set of particles alive at time t and $h_t(x)$ as the position of particle $x \in L_t$.

The study of extreme values of h dates back to works of Ikeda et al. [25, 26, 27], McKean [32], Bramson [10, 12] and Lalley and Sellke [28] who derived asymptotics for the law of the maximal height $h_t^* = \max_{t \in L_t} h_t(x)$. Introducing the centering function

$$m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log^+ t, \quad \text{where} \quad \log^+ t := \log(t \vee 1), \quad (1.1)$$

and writing \hat{h}_t for the centered process $h_t - m_t$ and \hat{h}_t^* for its maximum, they show that \hat{h}_t^* converges in law to $G + \log Z$ as $t \rightarrow \infty$, where G is a Gumbel distributed random variable and Z , which is independent of it, is the almost sure limit as $t \rightarrow \infty$ of (a multiple of) the so-called *derivative martingale*:

$$Z_t := C_\diamond \sum_{x \in L_t} (\sqrt{2}t - h_t(x)) e^{\sqrt{2}(h_t(x) - \sqrt{2}t)}, \quad (1.2)$$

for some $C_\diamond > 0$ properly chosen. (Henceforth we use this unconventional normalization, to avoid carrying the constant C_\diamond around in all occurrences of Z .)

Other extreme values of h can be studied simultaneously by considering the extremal process:

$$\mathcal{E}_t := \sum_{x \in L_t} \delta_{h_t(x) - m_t}. \quad (1.3)$$

Asymptotics for this process were treated in the physics literature by, e.g., Brunet and Derrida [13] and more recently in the mathematical literature simultaneously by Aïdékon et al. [2] and Arguin et al. [4]. These works show that there exists a random point measure \mathcal{E} such that

$$\mathcal{E}_t \Longrightarrow \mathcal{E} \quad \text{as } t \rightarrow \infty, \quad (1.4)$$

in the sense of weak convergence of distributions on the space \mathbb{M} of Radon measures on \mathbb{R} endowed with the vague topology. The process \mathcal{E} turns out to be a randomly shifted clustered Poisson point process with an exponential intensity. More explicitly, there exists a non-degenerate *cluster distribution* ν on the set of point measures in \mathbb{M} with support in $(-\infty, 0]$, such that \mathcal{E} can be realized as

$$\mathcal{E} := \sum_{k \geq 1} \mathcal{C}^k(\cdot - u^k) \quad \text{where} \quad \mathcal{E}^* = \sum_{k \geq 1} \delta_{u^k} \sim \text{PPP}(Ze^{-\sqrt{2}u} du), \quad (1.5)$$

Z is as before, $(\mathcal{C}^k : k \geq 1)$ are independently (of each other and of Z and \mathcal{E}^*) chosen according to ν and $(u^k : k \geq 1)$ enumerate the atoms of \mathcal{E}^* in decreasing order.

In fact, a slightly stronger version of the above convergence holds. To state it, let us first endow the set L_t with the genealogical distance $d = d_t$ given by

$$d(x, x') := \inf\{s \geq 0 : x, x' \text{ share a common ancestor in } L_{t-s}\}, \quad x, x' \in L_t, \quad t \geq 0. \quad (1.6)$$

Then, given $x \in L_t$ and $r > 0$, we let $\mathcal{C}_{t,r}(x)$ denote the (finite time, finite diameter) cluster of relative particle heights, at genealogical distance at most r from x , defined formally as

$$\mathcal{C}_{t,r}(x) := \sum_{y \in B_r(x)} \delta_{h_t(y) - h_t(x)}, \quad \text{where} \quad B_r(x) := \{y \in L_t : d(x, y) < r\}. \quad (1.7)$$

Finally, fixing any positive function $t \mapsto r_t$ such that both r_t and t/r_t tend to ∞ as $t \rightarrow \infty$ and letting $L_t^* = \{x \in L_t : h_t(x) \geq h_t(y), \forall y \in B_{r_t}\}$, we can define the *generalized extremal process* $\widehat{\mathcal{E}}_t$ as

$$\widehat{\mathcal{E}}_t := \sum_{x \in L_t^*} \delta_{h_t(x) - m_t} \otimes \delta_{\mathcal{C}_{t,r_t}(x)}. \quad (1.8)$$

The process $\widehat{\mathcal{E}}_t$, which is a random point measure on $\mathbb{R} \times \mathbb{M}$, records both the centered height of and the cluster around r_t -local maxima of h .

Then the proof of Theorem 2.3 in [4] readily shows that

$$(\widehat{\mathcal{E}}_t, Z_t) \xrightarrow{t \rightarrow \infty} (\widehat{\mathcal{E}}, Z), \quad \text{with} \quad \widehat{\mathcal{E}} \sim \text{PPP}(Ze^{-\sqrt{2}u} du \otimes \nu), \quad (1.9)$$

and Z_t, Z and ν as before. In fact, one can realize $\mathcal{E}, \mathcal{E}^*$ and $\widehat{\mathcal{E}}$ on the same probability space such that

$$\mathcal{E}^* = \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}} \delta_u \quad \text{and} \quad \mathcal{E} = \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}} \mathcal{C}(\cdot - u), \quad (1.10)$$

with the sums running over all points in the support of $\widehat{\mathcal{E}}$. Moreover, letting

$$\mathcal{E}_t^* := \sum_{x \in L_t^*} \delta_{h_t(x) - m_t}, \quad (1.11)$$

we clearly have $\mathcal{E}_t^* \Rightarrow \mathcal{E}^*$ as $t \rightarrow \infty$.

This explains the clustered structure of the limit process \mathcal{E} as given by (1.5). The “backbone” Poisson point process \mathcal{E}^* captures the asymptotics of extreme values which are also the local maxima in an $O(1)$ -genealogical neighborhoods around them, while the clusters $(\mathcal{C}^k : k \geq 1)$ describe the asymptotic law of the (relative) heights of particles in these neighborhoods. The validity of this description, or equivalently of relation (1.10) is a consequence of the following result from [3] (Theorem 2.1), which shows that particles achieving extreme height separate in the limit into clusters of diameter $O(1)$ which are $t - O(1)$ apart (in genealogical distance):

$$\sup_{v \in \mathbb{R}} \lim_{r \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\exists x, y \in L_t : h_t(x), h_t(y) > m_t + v \text{ and } r < d(x, y) < t - r) = 0. \quad (1.12)$$

1.2 Results

In this manuscript we provide a more detailed description of the extreme level sets of branching Brownian motion, improving upon the state-of-the-art as outlined above (see also Subsection 1.4). The term *extreme (super/upper) level set* will be used in this work to refer to the set of indices or heights of particles in L_t whose value under h_t is above $m_t + v$ for some fixed $v \in \mathbb{R}$. In light of convergence statements (1.4) and (1.9), such results can be stated, rather equivalently, both in an asymptotic form or directly in terms of the limiting objects. Since each form is of interest by itself, we will use both formulations.

In what follows, we say that $f(u, v)$ converges to F in the limit when $u \rightarrow u_0$ followed by $v \rightarrow v_0$, to mean that $\lim_{v \rightarrow v_0} \limsup_{u \rightarrow u_0} |f(u, v) - F| = 0$. If $f(u, v) = f_w(u, v)$ and $F = F_w$, then this convergence is uniform in $w \in \mathcal{W}$, if the above holds with an additional $\sup_{w \in \mathcal{W}}$ before the absolute value. We write $f(u) \sim g(u)$ as $u \rightarrow u_0$ to mean that $f(u)/g(u) \rightarrow 1$ as $u \rightarrow u_0$.

This should not be confused with the notation for “is distributed according to” which will use the same symbol. Finally, arbitrary positive constants are marked by decorated version of the letter C (e.g. C') and unless otherwise noted, change their value from one line to another.

1.2.1 Extreme Level Sets

Our first result concerns the asymptotic size of the level set of extreme values at height $m_t - v$. The following theorem confirms a conjecture by Brunet and Derrida (Subsection 4.3 in [13], see also Subsection 1.4 below).

Theorem 1.1. *There exists $C_\star > 0$ such that*

$$\frac{\mathcal{E}([-v, \infty))}{C_\star Z v e^{\sqrt{2}v}} \xrightarrow{\mathbb{P}} 1 \quad \text{as } v \rightarrow \infty. \quad (1.13)$$

In particular, for all $\epsilon > 0$,

$$\lim_{v \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{\mathcal{E}_t([-v, \infty))}{C_\star Z v e^{\sqrt{2}v}} - 1 \right| > \epsilon \right) = 0. \quad (1.14)$$

The asymptotic growth rate (as $v \rightarrow \infty$) of the number of points in \mathcal{E} should be compared with the growth rate of the number of points in the process \mathcal{E}^* , which records the limit of only those extreme values which are also local maxima. It follows from (1.5) and a simple application of the weak law of large numbers that

$$\frac{\mathcal{E}^*([-u, \infty))}{Z e^{\sqrt{2}u} / \sqrt{2}} \xrightarrow{\mathbb{P}} 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{\mathcal{E}_t^*([-u, \infty))}{Z e^{\sqrt{2}u} / \sqrt{2}} - 1 \right| > \epsilon \right) = 0. \quad (1.15)$$

The above shows that points coming from the clusters around extreme local maxima account for an additional multiplicative linear prefactor in the overall growth rate of extreme values.

This gives rise to the following natural question: What is the “typical” height of those local maxima in $\mathcal{E}_t^*|_{[-v, \infty)}$ whose cluster points “carry” the level set $\mathcal{E}_t|_{[-v, \infty)}$? As the next proposition shows, the contribution is essentially uniform across all heights in $[-v, \infty)$. For a precise statement, recall (1.10) and given a Borel set $B \subseteq \mathbb{R} \times \mathbb{M}$ define,

$$\mathcal{E}(\cdot; B) := \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}} \mathcal{C}(\cdot - u) 1_{\{(u, \mathcal{C}) \in B\}} \quad \text{and} \quad \mathcal{E}_t(\cdot; B) := \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}_t} \mathcal{C}(\cdot - u) 1_{\{(u, \mathcal{C}) \in B\}}. \quad (1.16)$$

Theorem 1.2. *Fix any $\alpha \in (0, 1]$. Then as $v \rightarrow \infty$,*

$$\frac{\mathcal{E}([-v, \infty); [-\alpha v, \infty) \times \mathbb{M})}{\mathcal{E}([-v, \infty))} \xrightarrow{\mathbb{P}} \alpha. \quad (1.17)$$

In particular,

$$\lim_{v \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{\mathcal{E}_t([-v, \infty); [-\alpha v, \infty) \times \mathbb{M})}{\mathcal{E}_t([-v, \infty))} - \alpha \right| > \epsilon \right) = 0. \quad (1.18)$$

We can rephrase the statement in (1.18) in terms of a uniform sampling from all particles whose height is above $m_t - v$ as follows:

Corollary 1.3. *Given $t, v > 0$, let X be a particle chosen uniformly from all particles $x \in L_t$ satisfying $\hat{h}_t(x) \geq -v$ and set $Y := \operatorname{argmax}\{\hat{h}_t(y) : y \in B_{t,r_t}(X)\}$. Then as $t \rightarrow \infty$ followed by $v \rightarrow \infty$,*

$$\frac{\hat{h}_t(Y) - (-v)}{v} \implies U([0, 1]). \quad (1.19)$$

Roughly speaking, for each $u \in [O(1), v]$ the total contribution to $\mathcal{E}_t([-v, \infty))$ from clusters around local maxima at height $m_t - u$ is uniformly $\sim C_* Z e^{\sqrt{2}v}$, making the total size of the level set $\sim C_* Z v e^{\sqrt{2}v}$ in agreement with Theorem 1.1.

Next we show that the contribution at each height $m_t - u$ (as discussed above) is carried by only a small $O((v - u)^{-1})$ fraction of the clusters. These are exceptionally large clusters, each contributing an order of $(v - u)e^{\sqrt{2}(v - u)}$ points to the level set $\{h_t \geq m_t - v\}$. To this end, if $v \geq 0$, $\alpha \in [0, 1]$ and $\kappa > 0$, we set

$$T_{\alpha, \kappa}(v) := \left\{ (-u, \mathcal{C}) \in \mathbb{R} \times \mathbb{M} : -u \in [-\alpha v, \infty), \mathcal{C}([-v - u, \infty)) \geq \frac{(v - u)}{\kappa} e^{\sqrt{2}(v - u)} \right\}, \quad (1.20)$$

for the set of clusters as described above. Given a Borel set $B \subseteq \mathbb{R} \times \mathbb{M}$ we also define in analog to (1.16),

$$\mathcal{E}^*(\cdot; B) := \sum_{(u, \mathcal{C}) \in \hat{\mathcal{E}}} \delta_v 1_{\{(u, \mathcal{C}) \in B\}} \quad \text{and} \quad \mathcal{E}_t^*(\cdot; B) := \sum_{(u, \mathcal{C}) \in \hat{\mathcal{E}}_t} \delta_v 1_{\{(u, \mathcal{C}) \in B\}}. \quad (1.21)$$

Then,

Theorem 1.4. *For any $\epsilon > 0$, there exist $\kappa > 0$, such that for all $\alpha \in (0, 1)$ with probability tending to 1 as $v \rightarrow \infty$,*

$$\frac{\mathcal{E}^*([- \alpha v, \infty); T_{\alpha, \kappa}(v))}{\mathcal{E}^*([- \alpha v, \infty))} < \frac{\kappa}{(1 - \alpha)v}, \quad (1.22)$$

and

$$\frac{\mathcal{E}([-v, \infty); T_{\alpha, \kappa}(v))}{\mathcal{E}([-v, \infty); [- \alpha v, \infty) \times \mathbb{M})} \geq 1 - \epsilon. \quad (1.23)$$

The same holds in the limit when $t \rightarrow \infty$ followed by $v \rightarrow \infty$ if we replace \mathcal{E} and \mathcal{E}^ by \mathcal{E}_t and \mathcal{E}_t^* respectively.*

Lastly, we find the rate of decay of the right tail of the distribution of the distance between the maximum and the second maximum under h_t , thereby confirming another conjecture of Brunet and Derrida (Subsection 4.2 in [13]). Setting $h_t^{*(2)} := \max\{h_t(x) : x \in L_t, h_t(x) < h_t^*\}$, we have

Theorem 1.5. *Let $v^1 > v^2 > \dots$ be the ordered atoms of \mathcal{E} . Then*

$$\lim_{w \rightarrow \infty} w^{-1} \log \mathbb{P}(v^1 - v^2 > w) = -(2 + \sqrt{2}). \quad (1.24)$$

In particular,

$$\lim_{w \rightarrow \infty} \limsup_{t \rightarrow \infty} \left| w^{-1} \log \mathbb{P}(h_t^* - h_t^{*(2)} > w) + (2 + \sqrt{2}) \right| = 0. \quad (1.25)$$

1.2.2 Cluster Level Sets

As evident by (1.5) or (1.9), (1.10), the key to obtaining the theorems above lies in understanding the cluster distribution ν . Thanks to a good control over the convergences in (1.5), (1.9) and the explicit description of \mathcal{E} and $\hat{\mathcal{E}}$, one can turn local asymptotic properties of clusters into global statements concerning these limit processes, and then to asymptotic results for the extreme level sets of h_t itself. In this subsection we therefore state the cluster law properties, which are used to derive the main theorems in this paper. These properties should be of independent interest.

The first proposition concerns the asymptotic mean number of cluster particles at height $-v$ or above, as well as an upper bound on its second moment. Recall that by definition and (1.9), if $\mathcal{C} \sim \nu$ then $\mathcal{C}([0, \infty)) = \mathcal{C}(\{0\}) = 1$ almost-surely.

Proposition 1.6. *Let $\mathcal{C} \sim \nu$. Then with $C_\star > 0$ as in Theorem 1.1,*

$$\mathbb{E}\mathcal{C}([-v, 0]) \sim C_\star e^{\sqrt{2}v} \text{ as } v \rightarrow \infty. \quad (1.26)$$

Moreover, there exists $C > 0$ such that for all $v \geq 0$,

$$\mathbb{E}[\mathcal{C}([-v, 0])^2] \leq C(v + 1)e^{2\sqrt{2}v}. \quad (1.27)$$

As surmised by the above upper bound, the number of points in \mathcal{C} lying in $[-v, 0]$ does not concentrate around its mean for large v . Indeed, the next proposition shows that the asymptotic mean on the right hand side of (1.26), is the result of an unlikely event of probability $O(v^{-1})$, in which the number of cluster particles above $-v$, is of unusually high order $ve^{\sqrt{2}v}$.

Proposition 1.7. *Let $\mathcal{C} \sim \nu$. Then for all $\epsilon > 0$, there exists $\kappa > 0$, such that for all $v \geq 0$,*

$$\mathbb{P}\left(\mathcal{C}([-v, 0]) > \frac{ve^{\sqrt{2}v}}{\kappa}\right) \leq \frac{\kappa}{v}, \quad (1.28)$$

and

$$\mathbb{E}\left(\mathcal{C}([-v, 0]); \mathcal{C}([-v, 0]) \leq \frac{ve^{\sqrt{2}v}}{\kappa}\right) \leq \epsilon e^{\sqrt{2}v}. \quad (1.29)$$

In the next proposition we find the rate of decay in the right tail of the distribution of the distance between the top two cluster particles.

Proposition 1.8. *Let $\mathcal{C} \sim \nu$. Then,*

$$\lim_{v \rightarrow \infty} v^{-1} \log \mathbb{P}(\mathcal{C}([-v, 0]) = 0) = -2. \quad (1.30)$$

1.3 Proof Outline

Let us give a brief outline of the proof of the main results in this paper. As mentioned before, the key ingredient in deriving results pertaining to the extremal landscape of the process is the study of the cluster distribution ν . Aside from the limit of the derivative martingale Z , whose effect is merely a global shift, all remaining ingredients in the definition of \mathcal{E} and $\widehat{\mathcal{E}}$ are explicit. Properties of the cluster law can therefore be translated via (1.5) or (1.9) and (1.10), to properties of \mathcal{E} and $\widehat{\mathcal{E}}$ and through convergences (1.4) and (1.9) into asymptotic properties of the statistics of extreme values of h .

1.3.1 Cluster Level Sets

The study of cluster law properties, which constitutes the core of the paper, begins by observing that the product structure of the intensity measure in (1.9) and indistinguishability of particles, imply that we could focus on the limiting law of the cluster around a uniformly chosen particle X_t in L_t , conditioned to be the global maximum at time t and having height, say, m_t . Tracing the trajectory of this distinguished particle backwards in time and accounting, via the spinal decomposition (Many-to-One Lemma, see Subsection 2.2), for the random genealogical structure, one sees a particle performing a standard Brownian motion $W = (W_s)_{s \geq 0}$ from m_t at time 0 to 0 at time t . This, so-called, *spine particle* gives birth at random Poissonian times (with accelerated rate 2, see Subsection 2.2) to independent standard branching Brownian motions, which then evolve back to time 0 and are conditioned to have their particles stay below m_t at this time. The cluster distribution at genealogical distance r around X_t is therefore determined by the relative heights of particles of those branching Brownian motions which branched off before time r (see Figure 1).

Denoting by $0 \leq \sigma_1 < \sigma_2 < \dots$ the points of a Poisson point process \mathcal{N} on \mathbb{R}_+ with rate 2 and letting $H = (h_t^s(x) : t \geq 0, x \in L_t^s)_{s \geq 0}$ be a collection of independent branching Brownian motions (with W, \mathcal{N} and H independent), the above formalizes as (Lemma 5.1):

$$\begin{aligned} \nu(\cdot) &= \lim_{t \rightarrow \infty} \mathbb{P}(\mathcal{C}_{t,r_t}(X_t) \in \cdot \mid h_t^* = h_t(X_t) = m_t) \\ &= \lim_{t \rightarrow \infty} \mathbb{P}_{0,m_t}^{t,0} \left(\sum_{\sigma_k \leq r} \sum_{x \in L_{\sigma_k}^{\sigma_k}} \delta_{m_t - h_{\sigma_k}^{\sigma_k}(x)}(\cdot - W_{\sigma_k}) \in \cdot \mid \max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} + h_{\sigma_k}^{\sigma_k^*}) \leq m_t \right), \end{aligned} \quad (1.31)$$

where r_t is as in (1.8), $h_t^* = \max_{x \in L_t} h_t(x)$ and we write $\mathbb{P}_{0,x}^{t,y}$ for the conditional probability measure $\mathbb{P}(\cdot \mid W_0 = x, W_t = y)$.

Since the law of W_s under $\mathbb{P}_{0,m_t}^{t,0}$ is the same as that of $W_s + m_t(1 - \frac{s}{t})$ under $\mathbb{P}_{0,0}^{t,0}$, introducing $\widehat{W}_{t,s} := W_s - \gamma_{t,s}$ with $\gamma_{t,s} := \log^+ s - \frac{s}{t} \log^+ t$, we may rewrite the above as (Lemma 3.2):

$$\nu(\cdot) = \lim_{t \rightarrow \infty} \mathbb{P}_{0,0}^{t,0} \left(\sum_{k: \sigma_k \in [0,r_t]} \mathcal{E}_{\sigma_k}^{\sigma_k}(\cdot - \widehat{W}_{t,\sigma_k}) \in \cdot \mid \max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right), \quad (1.32)$$

where \mathcal{E}_t^s is the extremal process associated with h_t^s and $\widehat{h}_t^s = h_t^s - m_t$. The triplet $(\widehat{W}, \mathcal{N}, H)$

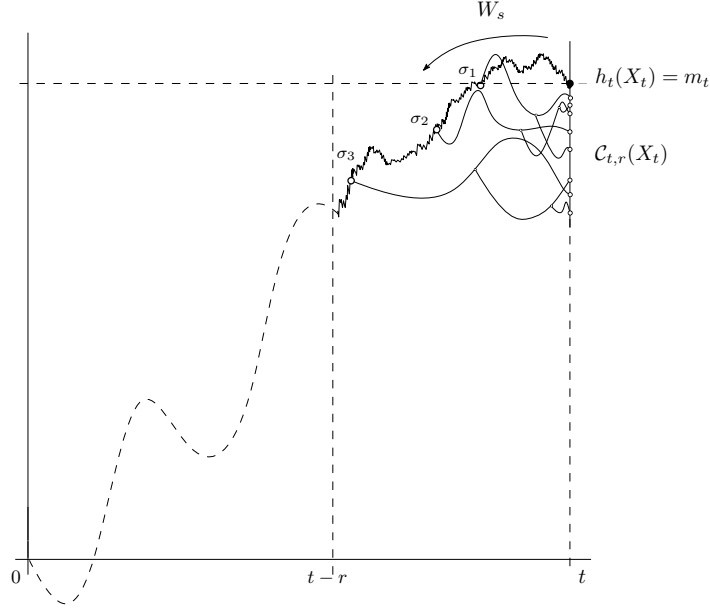


Figure 1: The cluster $\mathcal{C}_{t,r}(X_t)$ around the spine X_t , conditioned to be the maximum and at height m_t . The process W_s is a Brownian bridge from $(0, m_t)$ to $(t, 0)$ and $\sigma_1, \sigma_2, \dots$ are the branching times.

will be referred to as a *decorated random-walk-like* process (see Section 3). We remark that a similar characterization of the cluster distribution is given in [4].

The above representation can now be used to study the distribution of the size of cluster level sets as well as the law of the distance to the second highest particle in the cluster. To estimate the first moment of the size of the cluster level set, one can use (1.32), uniform integrability and Palm calculus to express $\mathbb{E}\mathcal{C}([-v, 0])$ for $\mathcal{C} \sim \nu$ and any $v \geq 0$ as the limit when $t \rightarrow \infty$ of

$$\frac{\int_0^{r_t} 2ds \int_{z=O(1)} \mathbb{E}\left(\mathcal{E}_s^s([-v, 0] - z); z + \widehat{h}_s^{s*} \leq 0\right) \mathbb{P}_{0,0}^{t,0}\left(\max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k*}) \leq 0, \widehat{W}_{t,s} \in dz\right)}{\mathbb{P}_{0,0}^{t,0}\left(\max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k*}) \leq 0\right)}. \quad (1.33)$$

Above, we have also conditioned on $\{\widehat{W}_{t,s} = z\}$ for $z = O(1)$ and used the total probability formula (see Lemma 5.4 and the proof of Lemma 5.2).

The left most term in the integrand is the first moment of the size of the (global) extreme level set of h_s^s , subject to a truncation event restricting the height of its global maximum. Using once again the spinal decomposition, we can express this expectation in terms of a probability involving (again) a uniformly chosen particle X_t as,

$$\begin{aligned} \mathbb{E}\left(\mathcal{E}_t([-v, u]); \widehat{h}_t^* \leq u\right) &= e^t \mathbb{P}(\widehat{h}_t(X_t) \in [-v, u], \widehat{h}_t^* \leq u) \\ &= e^t \int_{w=-v}^u \mathbb{P}(\widehat{h}_t^* \leq u \mid \widehat{h}_t(X_t) = w) \mathbb{P}(\widehat{h}_t(X_t) \in dw). \end{aligned} \quad (1.34)$$

where $v \leq 0$ and $u \geq -v$. As before, tracing the trajectory of the spine particle, the last conditional probability can be further expressed in terms of the decorated random-walk-like process as,

$$\mathbb{P}(\hat{h}_t^* \leq u \mid \hat{h}_t(X_t) = w) = \mathbb{P}_{0, w-u}^{t, -u} \left(\max_{k: \sigma_k \in [0, t]} (\widehat{W}_{t, \sigma_k} + \hat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right). \quad (1.35)$$

Examining (1.33) and (1.35), we see that to complete the derivation we need good estimates on probabilities of the form $\mathbb{P}_{0, x}^{t, y} \left(\max_{k: \sigma_k \in [0, t]} (\widehat{W}_{t, \sigma_k} + \hat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right)$, namely of the event that the random-walk-like process plus its decorations stays below 0 at random sampling times. For standard Brownian motion, the well known reflection principle gives

$$\mathbb{P}_{0, x}^{t, y} \left(\max_{s \in [0, t]} W_s \leq 0 \right) \sim \frac{2xy}{t} \quad \text{as } t \rightarrow \infty, \quad (1.36)$$

uniformly in $x, y \leq 0$ satisfying $xy = o(t)$ and with the right hand side holding as an upper bound for all $t \geq 0$ and $x, y \leq 0$. We show (Subsection 2.1) that similar estimates hold for the decorated random-walk-like process as well. This is not very surprising, as the drift function $\gamma_{t, s}$ is bounded by $1 + \log^+(s \wedge (t - s))$ (Lemma 3.3 with $r = 0$), the random decorations $(h_s^{s^*} : s \geq 0)$ are (at least) exponentially tight (Lemma 2.9) and the random sampling times $(\sigma_k : k \geq 1)$ arrive at a Poissonian rate.

Using such estimates in (1.35) one obtains $\mathbb{P}(\hat{h}_t^* \leq u \mid \hat{h}_t(X_t) = w) \approx C(u^+ + 1)(u - w)t^{-1}$ (in this section \approx means “roughly equals to”). This can then be used in (1.34) together with

$$\mathbb{P}(\hat{h}_t(X_t) \in dw) = \mathbb{P}(h_t(x) - m_t \in dw) = (2\pi t)^{-1/2} e^{-(m_t + w)^2/2t} \approx C t e^{-t} e^{\sqrt{2}w - w^2/(2t)} dw, \quad (1.37)$$

to yield (Lemma 4.2)

$$\mathbb{E}(\mathcal{E}_t([-v, u]); \hat{h}_t^* \leq u) \approx (u^+ + 1)(u + v) C e^{\sqrt{2}v - v^2/(2t)}. \quad (1.38)$$

Plugging this back into the integral in (1.33) and estimating the probability in the denominator by Ct^{-1} and the probability in the numerator by $Cz^2(s(t - s))^{-1} \mathbb{P}_{0, 0}^{t, 0}(\widehat{W}_{t, s} \in dz) \approx Ct^{-1}s^{-3/2}z^2$, one obtains (after integration over z),

$$\mathbb{E}\mathcal{C}([-v, 0]) \approx C v e^{\sqrt{2}v} \int_{s=0}^{\infty} s^{-3/2} e^{-v^2/(2s)} ds = C e^{\sqrt{2}v} \int_{r=0}^{\infty} r^{-3/2} e^{-1/(2r)} dr = C' e^{\sqrt{2}v}, \quad (1.39)$$

which is the first part of Proposition 1.6 with $C_\star = C'$. Similar computations, albeit more involved, can be used to obtain an upper bound on the second moment of $\mathcal{C}([-v, 0])$ as in the second part of Proposition 1.6.

Looking more carefully at (1.39), we see that for large v , “most” of the contribution to the first integral comes from $s \in [\eta v^2, \eta^{-1}v^2]$ (for “small” $\eta > 0$). Rolling back the derivation all the way to (1.33), the latter is equivalent to saying that most of the contribution to $\mathbb{E}([-v, 0])$ comes from trajectories in which $\max_{s \in [\eta v^2, \eta^{-1}v^2]} |\widehat{W}_{t, s}| = O(1)$. Such trajectories have very

small probability:

$$\mathbb{P}_{0,0}^{t,0} \left(\max_{s \in [\eta v^2, \eta^{-1} v^2]} |\widehat{W}_{t,s}| = O(1) \mid \max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right) \approx \int_{s=\eta v^2}^{v^2/\eta} C s^{-3/2} ds = C v^{-1}. \quad (1.40)$$

But when they occur, it follows from (1.38) with $s \in [\eta v^2, \eta^{-1} v^2]$ and an additional concentration argument that,

$$\mathcal{C}([-v, 0]) \approx C \mathcal{E}_s^s([-v, 0] - O(1)) 1_{\{\widehat{h}_s^{**} \leq O(1)\}} \approx C' v e^{\sqrt{2}v}. \quad (1.41)$$

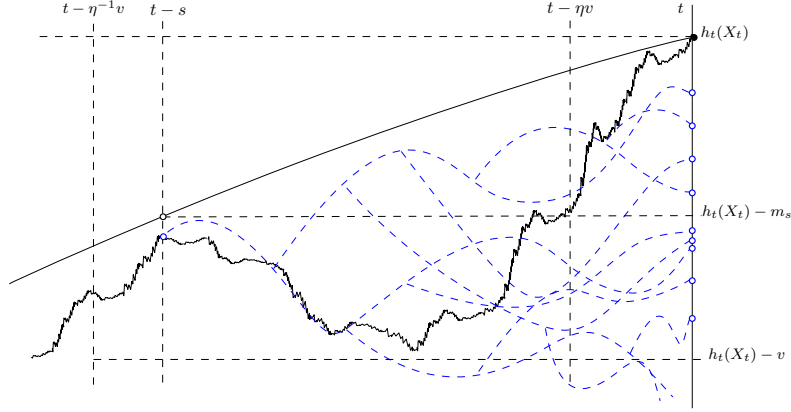


Figure 2: The contribution to the cluster level set $\mathcal{C}_{t,r_t}(X_t)|_{(-v,0]}$ on an atypical event when the cluster maximum X_t ascends from $h_t(X_t) - m_s + O(1)$ at time $t - s$ to $h_t(X_t)$ at time t , where $s \in [\eta v^2, \eta^{-1} v^2]$.

Reversing further reduction steps (1.31) and (1.32), we can rephrase the last statement as follows (Lemma 5.7): Having $Ce^{\sqrt{2}v}$ mean number of particles above $-v$ (for large v) in a cluster is the result of an atypical $O(v^{-1})$ probability event, in which the number of such particles is $Cve^{\sqrt{2}v}$. Such a cluster is realized at a large time t when its local maximum X_t has ascended (atypically slowly) to $h_t(X_t)$ from $h_t(X_t) - m_s + O(1)$ at time $t - s$, where $s \in [\eta v^2, \eta^{-1} v^2]$ (see Figure 2). This is the content of Proposition 1.7.

1.3.2 Extreme Level Sets

As suggested before, we can take advantage of convergences (1.4) and (1.9) to prove all results for the limit processes \mathcal{E} and $\widehat{\mathcal{E}}$ first and then convert these to asymptotic statements for h_t , using standard weak convergence arguments for random measures. Working directly with the limiting objects has the advantage that, equipped with the needed cluster properties, their law has an explicit and rather simple form (see (1.5), (1.9), (1.10)).

Let us demonstrate this by deriving asymptotics for the size of extreme level sets (Theorem 1.1). To this end, we aim to show that $\mathcal{E}([-v, \infty))v^{-1}e^{-\sqrt{2}v} \rightarrow C_* Z$ as $v \rightarrow \infty$ in probability (Lemma 6.1). Using (1.5), we can begin by writing $\mathcal{E}([-v, \infty))$ as the sum $\sum_{k \geq 1} \mathcal{C}^k([-v - u^k, 0])$,

with \mathcal{C}^k , u^k as in the display. Ignoring terms with $u^k \notin [-v + \sqrt{\log v}, \sqrt{\log v}]$, which are negligible in the scale we consider (see proof of Lemma 6.1) and denoting by $\tilde{\mathcal{E}}([-v, \infty))$ the sum of the remaining terms, we can condition on Z and use (1.39) together with the Poisson law of \mathcal{E}^* to estimate $\mathbb{E}(\tilde{\mathcal{E}}([-v, \infty)) \mid Z)$ by

$$\int_{-v+\sqrt{\log v}}^{\sqrt{\log v}} \mathbb{E}\mathcal{C}([-v-u, 0]) Z e^{-\sqrt{2}u} du \approx \int_{-v+\sqrt{\log v}}^{\sqrt{\log v}} C_* e^{\sqrt{2}(v+u)} Z e^{-\sqrt{2}u} du \approx C_* Z v e^{\sqrt{2}v}. \quad (1.42)$$

A similar computation using the second moment bound on $\mathbb{E}\mathcal{C}([-v-u, 0])$ in place of the first, shows that the conditional (on Z) variance of $\tilde{\mathcal{E}}([-v, \infty))$ is at most Cv^{-1} times its conditional mean. Then Chebyshev's inequality shows that $\tilde{\mathcal{E}}([-v, \infty))$ is concentrated around its conditional mean, which in light of (1.42) and $\tilde{\mathcal{E}}([-v, \infty)) \approx \mathcal{E}([-v, \infty))$ yields the desired result.

Other properties of the extreme level sets are equally easy to prove. For instance, to find asymptotics for $\mathcal{E}([-v, \infty); [-\alpha v, \infty) \times \mathbb{M})$ with $\alpha \in (0, 1)$, one merely changes the lower bound of the integral in (1.42) to $-\alpha v$. Then the integral evaluates to $C_* Z \alpha v e^{\sqrt{2}v}$ and since the quantity is still concentrated, combining this result with the previous one gives Theorem 1.2.

1.3.3 Distance to the Second Maximum

Lastly, let us discuss the upper tail decay of the law governing the distance between the first and second maxima of h , namely Theorem 1.5 and Proposition 1.8 on which the theorem relies. Again, thanks to the convergence of the extremal process, we can look at the distance between the two highest points $v^1 > v^2$ in \mathcal{E} . Then, the clustered structure of the limit (1.5) readily shows that these points are at least $w > 0$ apart, if and only if the distance between the two highest local maxima $u^1 > u^2$ in \mathcal{E}^* and the distance to the second highest particle in cluster \mathcal{C}^1 of u^1 are both at least w . Thanks to independence, we therefore get

$$\mathbb{P}(v^1 - v^2 > w) = \mathbb{P}(u^1 - u^2 > w) \mathbb{P}(\mathcal{C}([-w, 0]) = 0). \quad (1.43)$$

The first probability on the right hand side evaluates to $Ce^{-\sqrt{2}w}$ (see proof of Theorem 1.5). This is an easy exercise in Poisson point processes, after noticing that the random shift governing the law of \mathcal{E}^* can be just ignored.

For the second probability (Proposition 1.8), we again use the random-walk representation of the cluster distribution, per (1.31) and (1.32) and estimate instead the probability that $\mathcal{C}_{t,r_t}([w, 0]) = 0$ as $t \rightarrow \infty$ under the conditional measure, where $\mathcal{C}_{t,r_t} = \mathcal{C}_{t,r_t}(X_t)$. For a lower bound, we follow the heuristics of Brunet and Derrida (Subsection 4.2 in [13]) and observe that having no points in $\mathcal{C}_{t,r_t}|_{[-w,0]}$ can be realized by the intersection of the event that $\widehat{W}_{t,s}$ reaches height $-w$ or below at some time $s = \tau \in (0, r_t)$ without branching, with the event that $\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^* \leq -w$ for all $\sigma_k \in [\tau, r_t]$.

Now, the probability of the first event is, up to sub-exponential terms, $e^{-w^2/(2\tau)} \times e^{-2\tau}$. This is clearly the case without the conditioning, but can be shown to hold also under the conditional measures in (1.32). When $\widehat{W}_{t,\tau} \leq -w$, an entropic repulsion effect, which is the result

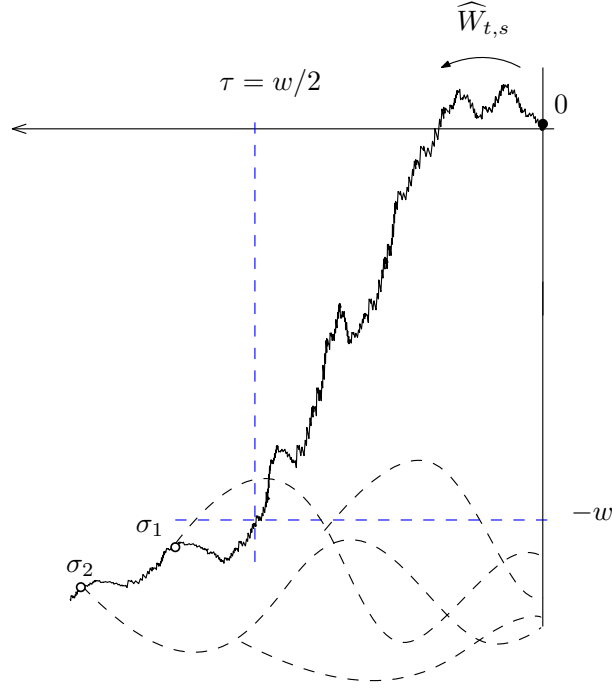


Figure 3: A typical realization for $\{\mathcal{C}([-w, 0)) = 0\}$: The process $\widehat{W}_{t,s}$ reaches height $-w$ at time $\tau = w/2$ without branching and then, along with its decorations, stays below $-w$ until time r_t .

of conditioning the random-walk-like process plus its decorations to stay negative, makes the probability of the second event decay only polynomially in w (uniformly in t). Multiplying the two yields $e^{-w^2/(2\tau)-2\tau}$ as a lower bound (on an exponential scale) on the conditional probability of $\{\mathcal{C}_{t,r_t}([-w, 0)) = 0\}$ for any choice of τ and all t large enough. The exponent is maximized at $\tau = w/2$, yielding a lower bound of e^{-2w} (see Figure 3).

A matching upper bound can be obtained by stopping the process $\widehat{W}_{t,s}$ at the first time T when it reaches height $-w(1-\epsilon)$ for $\epsilon > 0$. Then up to this time and if w is large, any branching event will result in violation of the condition $\mathcal{C}([-w, 0)) = 0$ with probability $1 - \delta$, where $\delta > 0$ can be made arbitrarily small, by choosing ϵ appropriately. This makes the probability of having no points in $[-w, 0)$ conditional on T at most $e^{-2(1-\delta)T}$ and gives an overall upper bound (on an exponential scale) of $e^{-w^2/(2\tau)-2(1-\delta)\tau}$ on the probability that $\mathcal{C}_{t,r_t}([-w, 0)) = 0$ and $T \in d\tau$, under the conditional measure in (1.32). Integrating with respect to τ , we are led to the maximization problem from before, and consequently obtain $e^{-(2-\delta')w}$ as an upper bound for all $t \geq 0$ and arbitrarily small $\delta' > 0$, as desired.

1.4 Context, Extensions and Open Problems

Branching Brownian motion is among the most fundamental random processes in modern probability theory. Aside from an intrinsic mathematical interest, the motivation for considering

such a model comes from various disciplines, such as biology, where it is a canonical choice for describing population dynamics (e.g. [20]) and physics, where, among other things, it can be used to model correlated energy levels in spin-glass-type systems [9, 18, 17]. In mathematics, it has deep connections with analysis, e.g. via the F-KPP equation (used by McKean [32] to derive asymptotics for the centered maximum) as well as other fields in probability such as random matrices [21], super-processes [16], multiplicative chaos [33] and more. We invite the reader to consult [6, 34] for recent sources on this and related models.

From the point of view of extreme value theory, results of the past few years have shown that branching Brownian motion belongs to the same universality class as other models, where correlations are “scale-free” (either logarithmic or tree-like). These include the branching random walk [1, 29], the two-dimensional Gaussian free field [5, 11] (and logarithmically correlated Gaussian fields in general [19]), characteristic polynomials of GUE ensembles [22] and more. In all of these models the asymptotic form of the extremal process (or at least the derived law of the centered maximum) is that of a randomly shifted clustered Poisson point process with an exponential intensity, as in (1.5), albeit with different laws for the shift and cluster decorations.

Statistics of extreme values of such systems are interesting for multiple reasons. From a pure-mathematical perspective, logarithmic or tree-like correlations can be thought of as the next natural step after the i.i.d. case, where the theory of extreme values is fully developed. More applicatively, the very large (or very small) values in a system often correspond to quantities of interest in the reality which the model describes. For instance, interpreting the heights as energy levels in a spin-glass system, (negative) extreme values capture the lowest energy states. The latter carry the corresponding Gibbs distribution at low temperature (glassy-phase) [14, 24, 30].

A more general definition of the process than the one given in Subsection 1.1 would allow any (square-integrable, mean larger than one) distribution for the number of offspring at a branching event, any branching rate and any speed for the underlying motion. These, of course, do not change the validity of the convergence of the extremal process nor its limit, at a qualitative level. Other variants of branching Brownian motion, of which extreme values have been studied, include ones where the speed of particles changes with time [8] and where particle heights are complex-valued [24, 30]. Genealogical information concerning the extreme values of the process, both at a microscopic [31] and macroscopic [7] scales, can be described by considering properly generalized versions of the extremal process \mathcal{E}_t .

Getting back to our results, the extension to branching Brownian motion with a general offspring distribution requires only minor changes in the proofs. For simplicity, we treated the binary splitting case only. All theorems and propositions will therefore still hold, albeit with different constants. In fact, the proofs should also carry over without any conceptual difficulty to the branching random walk and the two-dimensional Gaussian free field. This is because the three main ingredients in the proofs (see Subsection 1.3): convergence of the extremal process, random walk representation of the cluster distribution and uniform tails for the centered maximum, are available in these two models as well. Consequently we expect all results, except for Theorem 1.5 and Proposition 1.8, to carry over (again with different constants and with the corresponding versions of the derivative martingale).

The statement of Proposition 1.8 depends crucially on the distribution of the difference between the heights of two nearby particles (in genealogical distance) or vertices (in lattice distance). Unlike for branching Brownian motion, where this difference can be made large by a delayed branching event, costing only an exponentially decaying probability (see Sub-Subsection 1.3.3), the tail of this difference is Gaussian for both the branching random walk and the Gaussian free field. This will result in a Gaussian decay for the probability in the statement of Proposition 1.8 and consequently also for the probability in Theorem 1.5, in these two models.

Finally, it should be noted that the statement in Theorem 1.1 does not translate into the existence of an asymptotic *spacing* between succeeding particles at extreme heights. Brunet and Derrida conjecture that the difference in height between the k -th and $k + 1$ -st highest particles is asymptotically $1/k - 1/(k \log k)$ as $k \rightarrow \infty$ (Subsection 4.1 in [13]). Claiming such a result, however, requires better control on the regularity of $\mathcal{E}([-v, \infty))$ and is left as an open problem.

Organization of the Paper

The remainder of the paper is organized as follows. Section 2 includes the necessary technical tools to be used in the proofs thereafter. These include mainly the random walk estimates discussed above as well as the spinal decomposition and uniform bounds on the tail of the centered maximum. In Section 3 we present the reduction statements, in which events concerning a spine particle are converted to events involving the decorated random-walk-like process. This section includes also some estimates for probabilities of such events, the proof of which uses the random-walk results from Section 2. Next comes Section 4, in which we use the reduction statements and the random-walk estimates to compute moments of $\mathcal{E}([-v, \infty))$ subject to a truncation event restricting the height of the global maximum. These in turn are used in Section 5 to derive all results concerning cluster level sets, i.e. all propositions in Subsection 1.2.2. Section 6 contains the proofs of all the theorems in Subsection 1.2.1, namely all extreme level set statements. Finally, Section 7 contains the proofs of the random walk statements from Subsection 2.1. These were left out from that subsection, since they are quite long and involved.

2 Technical Tools

In this section we introduce several technical tools which will be used throughout in the proofs to follow. Subsection 2.1 includes estimates on the probability that a random-walk-like process, with random time steps and decorations, stays below a curve. As explained in the proof outline (Subsection 1.3), such a process arises after various reduction steps, by tracing, backwards in time, a uniformly chosen particle reaching an extreme height. Because of the randomness of the underlying branching structure, the genealogy as seen from the point of view of this distinguished (spine) particle has a biased distribution. Spinal decomposition theory can then be used to account for this bias and to convert statements involving the spine particle to ones which pertain to all particles. This is the subject of Subsection 2.2. Finally Subsection 2.3 includes uniform bounds on the tail probabilities of the centered maximum.

Although the “random-walk” statements in Subsection 2.1 are standard in flavor, the particularity of the random-walk-like process to which they apply, implies that one cannot find them “on-the-shelf” and new proofs have to be provided. Since these are quite lengthy and technical (and perhaps not so interesting) we relegated them to Section 7.

2.1 Random Walk Estimates

For what follows, let $W = (W_u : u \geq 0)$ be a standard one dimensional Brownian motion. Given $0 \leq s < t$ and $x, y \in \mathbb{R}$, we shall denote by $\mathbb{P}_{s,x}^{t,y}$ and $\mathbb{P}_{s,x}$ the conditional distribution $\mathbb{P}(\cdot | W_s = x, W_t = y)$ and $\mathbb{P}(\cdot | W_s = x)$ respectively (if $s = 0$ we assume that W_0 was x in the first place). On the same probability space, let us suppose also the existence of an independent collection $Y = (Y_u : u \geq 0)$ of random variables, which is also independent of W . These random variables satisfy

$$\forall u, z \geq 0 : \mathbb{P}(|Y_u| \geq z) \leq \delta^{-1} e^{-\delta z} \quad (2.1)$$

for some $\delta \in (0, 1]$. The third collection of random variables defined on this space, comes in the form of a Poisson point process on \mathbb{R} :

$$\mathcal{N} \sim \text{PPP}(\lambda dx), \quad (2.2)$$

for some $\lambda > 0$. This process is assumed to be independent of W and Y and we denote by $(\sigma_k : k \geq 1)$ the collections of all atoms of \mathcal{N} , enumerated in increasing order. Given $s, t \geq 0$, let us write

$$\wedge^t(s) \equiv s \wedge (t - s). \quad (2.3)$$

We are interested in controlling the probability that the process $(W_{\sigma_k})_{\sigma_k \in \mathcal{N}}$ stays below the (perturbed) logarithmic curve $\delta^{-1}(1 + \log^+ \wedge^t(\sigma_k)) - Y_{\sigma_k}$ for all $\sigma_k \in [0, t]$. As an upper bound, we have

Proposition 2.1. *Suppose that W, Y, \mathcal{N} are defined as above with respect to some $\lambda, \delta > 0$. Then there exists $C = C(\lambda, \delta)$ such that for all $t \geq 0$, $x, y \in \mathbb{R}$,*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \delta^{-1}(1 + \log^+ \wedge^t(\sigma_k)) - Y_{\sigma_k}) \leq 0 \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t}, \quad (2.4)$$

Moreover, there exists $C' = C'(\lambda, \delta)$ such that for all $t \geq 0$ and all $x, y \in \mathbb{R}$ such that $xy \leq 0$,

$$\begin{aligned} \mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \delta^{-1}(1 + \log^+ \wedge^t(\sigma_k)) - Y_{\sigma_k}) \leq 0 \right) \\ \leq C' \frac{(x^- + e^{-\sqrt{2\lambda}(1-\delta)x^+})(y^- + e^{-\sqrt{2\lambda}(1-\delta)y^+})}{t} \exp \left(\frac{(y-x)^2}{2t} \right). \end{aligned} \quad (2.5)$$

For an asymptotic statement, we need two more ingredients. The first one is an additional assumption on the collection Y , in the form

$$Y_u \xrightarrow{u \rightarrow \infty} Y_\infty, \quad (2.6)$$

for some random variable Y_∞ . The second is the definition of a function $\gamma = \gamma_{t,u}$ from \mathbb{R}^2 to \mathbb{R} such that

$$-\delta^{-1} \leq \gamma_{t,u} \leq \delta^{-1}(1 + \log^+ \wedge^t(u)) , \quad 0 \leq u \leq t , \quad (2.7)$$

Finally, for all $u \geq 0$,

$$\gamma_{t,u} \xrightarrow{t \rightarrow \infty} \gamma_{\infty,u} , \quad \gamma_{t,t-u} \xrightarrow{t \rightarrow \infty} \gamma_{\infty,-u} , \quad (2.8)$$

where $\gamma_{\infty,u}, \gamma_{\infty,-u} \in \mathbb{R}_+$. We then have

Proposition 2.2. *Suppose that W, Y, \mathcal{N} and γ are defined as above and satisfy (2.1), (2.2), (2.6), (2.7) and (2.8) with respect to some $\delta, \lambda > 0$. Then there exists non-increasing positive functions $f, g : \mathbb{R} \rightarrow (0, \infty)$ depending on δ, λ, γ and Y , such that*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k}) \leq 0 \right) \sim 2 \frac{f(x)g(y)}{t} \quad \text{as } t \rightarrow \infty , \quad (2.9)$$

uniformly in x, y satisfying $x, y \leq 1/\epsilon$ and $(x^- + 1)(y^- + 1) \leq t^{1-\epsilon}$, for any fixed $\epsilon > 0$. Moreover,

$$\lim_{x \rightarrow \infty} \frac{f(-x)}{x} = \lim_{y \rightarrow \infty} \frac{g(-y)}{y} = 1 . \quad (2.10)$$

Remark 2.3 (Monotonicity with respect to initial conditions). *If $x, y, x', y' \in \mathbb{R}$ satisfy $x \leq x'$ and $y \leq y'$ and $t \geq 0$*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k}) \leq 0 \right) \geq \mathbb{P}_{0,x'}^{t,y'} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k}) \leq 0 \right). \quad (2.11)$$

Indeed, one can pass from a Brownian bridge from x to y to a Brownian bridge from x' to y' replacing W_s by $W_s - \left(\frac{s}{t}(y' - y) + (x' - x)(1 - \frac{s}{t}) \right)$ inside the probability brackets. Since the above interpolation function is positive for every $s \in [0, t]$ we can simply lower bound it by zero to obtain (2.11). In particular, it is straightforward to show that if the convergence from Proposition 2.2 holds, then both f and g are non-increasing.

We also need to know that the above asymptotics is continuous (in the sense specified below) in Y and γ . To this end for each $r \geq 0$, let $Y^{(r)}$ be a collection of random variables as Y above and $\gamma^{(r)}$ be a function as γ above, satisfying (2.1) and (2.7) uniformly for all $r \geq 0$ with $\delta > 0$. Suppose that (2.6) holds for $Y_u^{(r)}$ with the limit denoted by $Y_\infty^{(r)}$ and that (2.8) holds with the limits denoted by $\gamma_{\infty,u}^{(r)}$ and $\gamma_{-\infty,u}^{(r)}$. Then

Proposition 2.4. *Suppose that $W, Y, \mathcal{N}, \gamma$ and $Y^{(r)}, \gamma^{(r)}$ for $r \geq 0$ are defined as above with respect to some $\delta, \lambda > 0$. Let $f^{(r)}, g^{(r)}$ be the functions f, g respectively given in Proposition 2.2 applied to $W, Y^{(r)}, \mathcal{N}$ and $\gamma^{(r)}$. If for all $u \geq 0$,*

$$Y_\infty^{(r)} \xrightarrow{r \rightarrow \infty} Y_\infty , \quad Y_u^{(r)} \xrightarrow{r \rightarrow \infty} Y_u , \quad \gamma_{\infty,u}^{(r)} \xrightarrow{r \rightarrow \infty} \gamma_{\infty,u} , \quad \gamma_{-\infty,u}^{(r)} \xrightarrow{r \rightarrow \infty} \gamma_{-\infty,u} . \quad (2.12)$$

then for all $x \in \mathbb{R}$

$$f^{(r)}(x) \xrightarrow{r \rightarrow \infty} f(x) , \quad g^{(r)}(x) \xrightarrow{r \rightarrow \infty} g(x) , \quad (2.13)$$

with f, g given by Proposition 2.2 applied to W, Y, \mathcal{N} and γ . In particular if $Y_\infty^{(r)} = Y_\infty$ and $\gamma_{-\infty,u}^{(r)} = \gamma_{-\infty,u}$ for all $r \geq 0$ and $u \geq 0$, then $g^{(r)}(x) = g(x)$ for all $r \geq 0$.

Finally, we need a sharp entropic-repulsion type result. The sharpness of the statement necessitates some additional regularity conditions for γ . For our purposes it will suffice to assume that (see Figure 4):

$$\gamma_{t,u} - \frac{u}{r}\gamma_{t,r} \leq \delta^{-1}(1 + \log^+ \wedge^r(u)), \quad \gamma_{t,u'} - \frac{t-u'}{t-r}\gamma_{t,r} \leq \delta^{-1}(1 + \log^+ \wedge^{t-r}(u'-r)), \quad (2.14)$$

for all $0 < u < r < u' < t$. Under this condition we then have,

Proposition 2.5. *Suppose that W, Y, \mathcal{N} and γ are defined as above with respect to some $\delta, \lambda > 0$. Then for all $M \geq 1$ and $x, y \in \mathbb{R}$,*

$$\sup_{s \geq 1} \limsup_{t \rightarrow \infty} \sqrt{s} \mathbb{P}_{0,x}^{t,y} \left(\max_{[s,t-s]} W_u - \gamma_{t,u} \geq -M \mid \max_{\sigma_k \in [0,t]} (W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k}) \leq 0 \right) < \infty. \quad (2.15)$$

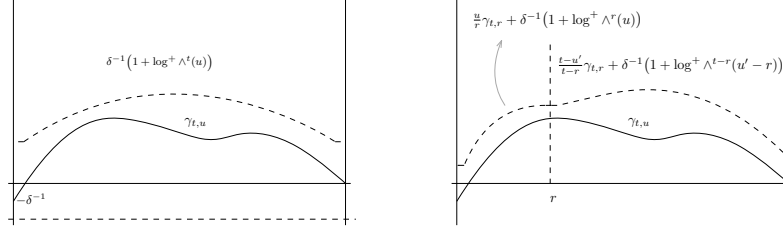


Figure 4: Illustrations of boundedness condition (2.7) (left) and regularity condition (2.14) (right), imposed on $u \mapsto \gamma_{t,u}$ as assumptions in Proposition 2.5. Only (2.7) is assumed in Proposition 2.4.

2.2 Spinal Decomposition

A key tool for reducing the computation of moments of the number of particles satisfying a certain condition is the so-call spinal decomposition, in the form of the two lemmas below. We refer the reader to [23] for a more general and thorough treatment of this method, as well as an historical overview.

For integer $k \geq 1$, the k -spine branching Brownian motion describes particles which branch and diffuse as in the original process, only that in addition they may carry “marks” indexed by the set $\{1, \dots, k\}$, which affect their branching and/or diffusion laws. For our purposes, we can assume that the diffusion law is always that of a standard Brownian motion and splitting is always binary, regardless of the carried marks. What is affected by the marks, is the branching rate, which is 2^m if the particle carries m marks. In addition, once a particle branches, each mark is transferred to one of its two children with equal probability and independently of the other marks.

As before, the set of particles at time t will be denoted by L_t , which again we equip with the genealogical metric $d = d_t$. The positions of particles will be given by the random collection $h_t = (h_t(x) : x \in L_t)$, again exactly as before. The new information, namely the location of the

marks at time t , will be denoted by the collection $X_t = (X_t(l) : l = 1, \dots, k)$, where $X_t(l) \in L_t$ is the particle holding mark l at time t . The genealogical line of decent of particle $X_t(l)$, namely the function $t \mapsto X_t(l)$, will be referred to as the l -th spine of the process.

We shall denote by $\tilde{\mathbb{P}}^{(k)}$ the underlying probability measure and by $\tilde{\mathbb{E}}^{(k)}$ the corresponding expectation. To simplify the notation in the case $k = 1$, we shall write $\tilde{\mathbb{P}}$, $\tilde{\mathbb{E}}$ and X_t in place of $\tilde{\mathbb{P}}^{(1)}$, $\tilde{\mathbb{E}}^{(1)}$ and $X_t(1)$. Note that in the case $k = 0$ the process is reduced to a regular branching Brownian motion, in which case we will keep using the notation \mathbb{P} , \mathbb{E} and use $(\mathcal{F}_t : t \geq 0)$ to denote its natural filtration.

The first lemma shows how to reduce first moment computations for regular branching Brownian motion to expectations involving the 1-spine measure. To avoid integrability issues, we state it for a bounded function, although this is entirely not necessary.

Lemma 2.6 (Many-to-one lemma). *Let $F = (F(x) : x \in L_t)$ be a bounded \mathcal{F}_t -measurable real-valued random function on L_t . Then,*

$$\mathbb{E}\left(\sum_{x \in L_t} F(x)\right) = e^t \tilde{\mathbb{E}}F(X_t). \quad (2.16)$$

The second lemma is suitable for second moment computations.

Lemma 2.7 (Many-to-two lemma). *Let $F = (F(x, y) : x, y \in L_t)$ be a bounded \mathcal{F}_t -measurable real-valued random function on $L_t \times L_t$. Then,*

$$\mathbb{E}\left(\sum_{x, y \in L_t} F(x, y)\right) = e^{3t} \tilde{\mathbb{E}}^{(2)}\left(F(X_t(1), X_t(2)) e^{-d(X_t(1), X_t(2))}\right). \quad (2.17)$$

Remark 2.8. *Observe that on the event $\{d(X_t(1), X_t(2)) = r\}$ for some $0 \leq r \leq t$, at all branching events prior to time $t - r$, which occur at rate 4, both spine particles “chose” to follow the same child. Since such events have probability $1/2$ and they are independent of each other, standard Poisson thinning arguments show that conditional on $\{d(X_t(1), X_t(2)) = r\}$ branching along the line of descent of the two spine particles up to time $t - r$ occurs at rate 2. Since the motion is not effected by the conditioning, we see that under the conditioning, the two-spine process behaves as a one-spine process up to time $t - r$, with the two spine particles identified. The same reasoning also implies that $(t - d(X_t(1), X_t(2))) \stackrel{\text{law}}{=} e \wedge t$, where e is an exponential random variable with rate 2.*

2.3 Uniform Tail Estimates for the Centered Maximum

Although asymptotics for the upper tail are well known, precise asymptotics for the lower tail are harder to find. Recall that we are writing h_t^* for $\max_{x \in L_t} h_t(x)$.

Lemma 2.9. *There exists $C, C' > 0$, such that for all $t \geq 0$ and $u \geq 0$,*

$$\mathbb{P}(h_t^* - m_t > u) \leq C u e^{-\sqrt{2}u} \quad \text{and} \quad \mathbb{P}(h_t^* - m_t < -u) \leq C' e^{-(2-\sqrt{2})u}. \quad (2.18)$$

Proof. Let $u(t, x) := \mathbb{P}(h_t^* > x)$ and $\tilde{m}_t := \inf\{x : u(t, x) \geq 1/2\}$, then it is well known [10] that $u(t, x)$ is the solution of the F-KPP equation

$$\partial_t u = \frac{1}{2} \partial_{xx} u + u(1 - u), \quad (2.19)$$

with initial data $u(0, x) = 1_{(-\infty, 0)}(x)$. Now Propositions 8.1 and 8.2 in [10] show that $|m_t - \tilde{m}_t| < C$ for all $t \geq 0$ and some $C > 0$, while Lemma 4.7 in [4], gives

$$u(t, x + \tilde{m}_t) \leq Cx \exp\left(-\sqrt{2}x - \frac{x^2}{2t} + \frac{3x}{2\sqrt{2}} \frac{\log t}{t}\right), \quad (2.20)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Combining the above gives the right-tail bound.

As for the left tail, Corollary 1 in [10] says that $u(t, \tilde{m}_t + x)$ is monotone decreasing in t for all $x < 0$ and that it converges as $t \rightarrow \infty$ to the traveling wave solution $\omega(x)$ of the F-KPP equation. The latter is the unique solution up to translations of the partial differential equation

$$\frac{1}{2}\omega' - \sqrt{2}\omega'' + \omega(1 - \omega) = 0. \quad (2.21)$$

An explicit computation of the left tail behavior of $1 - \omega(x)$ may be found in Appendix A of the [v1] arXiv version of [3] ($\omega(x)$ in their notation). The authors show that $1 - \omega(-x) \sim e^{-(2-\sqrt{2})x}$ as $x \rightarrow \infty$. Consequently for x large enough,

$$\begin{aligned} \mathbb{P}(h_t^* - m_t \leq -x) &\leq \mathbb{P}(h_t^* \leq \tilde{m}_t - x + C) = 1 - u(t, \tilde{m}_t - x + C) \\ &\leq 1 - \omega(-x + C) \leq C'e^{-(2-\sqrt{2})x}, \end{aligned} \quad (2.22)$$

as $x \rightarrow \infty$. This shows the bound for the left tail and completes the proof. \square

3 Reduction to a Decorated Random-Walk-Like Process

In the sequel we shall need to estimate probabilities concerning the height of one or two spine particles and the clusters around them, subject to a restriction on the global maximum of the process. By tracing the spine particles backwards in time, such events can be recast in terms of a decorated random-walk-like process, for which asymptotic probabilities are given in Subsection 2.1. We therefore proceed by defining this process explicitly and then stating various reduction lemmas which will be needed in the sequel. The section concludes with a few lemmas in which the probability of events involving the decorated process are estimated. These estimates will be used frequently in the proof to follow.

3.1 Definition of the Walk and Reduction Statements

As before let $W = (W_s : s \geq 0)$ be a standard Brownian motion, whose initial position we leave free to be determined according to the conditional statements we make. For $0 \leq s \leq t$, we fix

$$\gamma_{t,s} := \log^+ s - \frac{s}{t} \log^+ t \quad \text{and} \quad \widehat{W}_{t,s} := W_s - \gamma_{t,s}. \quad (3.1)$$

We shall also need the collection $H = (h^s = (h_t^s)_{t \geq 0} : s \geq 0)$ of independent copies of h , that we will assume to be independent of W as well. Finally, let \mathcal{N} be a Poisson point process with intensity $2dx$ on \mathbb{R}_+ , independent of H and W and denote by $\sigma_1 < \sigma_2 < \dots$ its ordered atoms. The triplet $(\widehat{W}, \mathcal{N}, H)$ forms the decorated random-walk-like process, which was eluded to in the beginning.

To see the relevance of the above process, recall that $B_r(x)$ is the ball of radius r around x in the genealogical distance d , and that we write $\widehat{h}_t = h_t - m_t$ and $\widehat{h}_t^* = \max_{x \in L_t} \widehat{h}_t(x)$. For $A \subseteq L_t$ set also $\widehat{h}_t^*(A)$ for $\max_{x \in A} \widehat{h}_t(x)$, then,

Lemma 3.1. *For all $0 \leq r \leq t$ and $u, w \in \mathbb{R}$,*

$$\widetilde{\mathbb{P}}\left(\widehat{h}_t^*(B_r^c(X(t))) \leq u \mid \widehat{h}_t(X_t) = w\right) = \mathbb{P}\left(\max_{k: \sigma_k \in [r, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t, r} = w - u, \widehat{W}_{t, t} = -u\right). \quad (3.2)$$

In particular for all $t \geq 0$ and $v, w \in \mathbb{R}$,

$$\widetilde{\mathbb{P}}(\widehat{h}_t^* \leq u \mid \widehat{h}_t(X_t) = w) = \mathbb{P}\left(\max_{k: \sigma_k \in [0, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t, 0} = w - u, \widehat{W}_{t, t} = -u\right). \quad (3.3)$$

Proof. Since both Brownian motion and Poisson point process are distributional invariant under time reversal, tracing the spine particle backwards in time, the left hand side of (3.2) can be written as

$$\mathbb{P}\left(\max_{k: \sigma_k \in [r, t]} (W_{\sigma_k} + h_{\sigma_k}^{\sigma_k^*}) \leq m_t + u \mid W_0 = m_t + w, W_t = 0\right). \quad (3.4)$$

where W_s , σ_k and h_t^σ are as above.

Now independence of \mathcal{N} , W and H together with standard Gaussian properties enjoyed by W imply that the probability above does not change if we replace W_s by $W_s + u + m_t(t - s)/t$ everywhere in (3.4). Replacing h_s^s and W_s by $\widehat{h}_s^s + m_s$ and by $\widehat{W}_{t, s} + \gamma_{t, s}$ respectively and observing that $m_t s/t - m_s = \gamma_{t, s}$, we obtain (3.2), then (3.3) follows by plugging in $r = 0$. \square

In a similar way, we can express the distribution of the cluster around the spine particle, given that it reaches height m_t . For what follows \mathcal{E}_t^s denotes the extremal process of h_t^s , defined as in (1.3) only with respect to h_t^s in place of h_t .

Lemma 3.2. *Let $\mathcal{A}_t := \left\{ \max_{k: \sigma_k \in [0, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right\}$, then for all $0 \leq r \leq t$ we have that*

$$\begin{aligned} & \widetilde{\mathbb{P}}\left((\mathcal{C}_{t, r}(X_t), (h_{t-s}(X_{t-s}) - m_t)_{s \in [0, r]}) \in \cdot \mid \widehat{h}_t^* = \widehat{h}_t(X_t) = 0\right) \\ &= \mathbb{P}\left(\left(\sum_{k: \sigma_k \in [0, r]} \mathcal{E}_{\sigma_k}^{\sigma_k^*}(\cdot - \widehat{W}_{t, \sigma_k}), (\widehat{W}_{t, s} - m_s)_{s \in [0, r]}\right) \in \cdot \mid \widehat{W}_{t, 0} = \widehat{W}_{t, t} = 0; \mathcal{A}_t\right). \end{aligned} \quad (3.5)$$

Proof. As in the proof of Lemma 3.1, we can replace $h_{t-s}(X_{t-s})$ in the left hand side of (3.5) by W_s , so that the conditioning event in the left hand side of (3.5) reads

$$\{W_0 = m_t, W_t = 0, \max_{k: \sigma_k \in [0, t]} W_{\sigma_k} + h_{\sigma_k}^{\sigma_k^*} - m_t \leq 0\}, \quad (3.6)$$

and $\mathcal{C}_{t,r}(X_t) \stackrel{\text{law}}{=} \sum_{x \in L_{\sigma_k}^{\sigma_k}} \delta_{W_{\sigma_k} + h_{\sigma_k}^{\sigma_k}(x) - m_t}$. The result follows after applying the same transformations as in the proof of Lemma 3.1. \square

The advantage of the above formulation, which uses the decorated random walk $\widehat{W}_{t,s}$, is that it is suitable for an application of the random walk estimates from Subsection 2.1, provided that $\gamma_{t,s}$ from (3.1) and \widehat{h}_s^{s*} satisfy the required conditions. Lemma 2.9 shows that \widehat{h}_s^{s*} satisfies the tail conditions with $\delta < (2 - \sqrt{2})^{-1}$. To check the conditions for $\gamma_{t,s}$, we shall need the following lemma.

Lemma 3.3. *Let $s, r, t \in \mathbb{R}$ be such that $0 \leq r \leq r + s \leq t$, then*

$$-1 \leq \log^+(r + s) - \left(\frac{t - (r + s)}{t - r} \log^+ r + \frac{s}{t - r} \log^+ t \right) \leq 1 + \log^+(s \wedge (t - r - s)). \quad (3.7)$$

Proof. Starting with the lower bound, it follows from the concavity of \log that 0 is lower bound when $r \geq 1$. If $r < 1$ and $r + s < 1$, then the middle expression is equal to $-s(\log^+ t)/(t - r)$, which is again greater than -1 . Lastly if $r < 1$ but $r + s \geq 1$, then the middle expression is equal to $\log(r + s) - s(\log t)/(t - r)$, whose minimum, attained at $s = 1 - r$, is again greater than -1 .

For the upper-bound, we consider the two cases $s \leq (t - r)/2$ and $s > (t - r)/2$ separately. In the first case, by replacing $\log^+ t$ by $\log^+ r$ in the middle expression, it is enough to prove the upper bound for $\log^+(r + s) - \log^+ r$. But, concavity of \log implies that the latter is at most $1 + \log^+ s$, which proves the statement for $s \leq (t - r)/2$. On the other hand, if $s > (t - r)/2$ we set $s' = t - r - s$ and rewrite the middle expression in (3.7) as

$$\log^+(t - s') - \left(\frac{s'}{t - r} \log^+ r + \frac{t - r - s'}{t - r} \log^+ t \right) \leq (\log^+ t - \log^+ r) \frac{s'}{t - r}. \quad (3.8)$$

Above, to get the second inequality, we have bounded $\log^+(t - s')$ by $\log^+ t$. Appealing to concavity of the logarithm function again, if $r \geq 1$, then the right hand side above is further upper bounded by $\log(s' + r) - \log r$ which is again smaller than $1 + \log^+ s'$ as before, which is what we need to show in this case. If $r < 1$ and $t < 1$, then the upper bound is trivial. Finally, if $r < 1$ and $t \geq 1$, then the upper bound follows from the inequality $s' \log t \leq (t - 1)(1 + \log^+ s')$ which holds for all $s' \leq t$. \square

3.2 Fundamental Estimates

With the above result at hand, we can state the following two lemmas, which are essentially corollaries of the random walk estimates from Subsection 2.1. In the first one, we obtain upper bounds and asymptotics for the probabilities appearing in Lemma 3.1.

Lemma 3.4. *There exists $C, C' > 0$ such that for all $0 \leq r \leq t$ and $w, v \in \mathbb{R}$,*

$$\mathbb{P} \left(\max_{k: \sigma_k \in [r, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k*}) \leq 0 \mid \widehat{W}_{t, r} = v, \widehat{W}_{t, t} = w \right) \leq C \frac{(v^- + 1)(w^- + 1)}{t - r}. \quad (3.9)$$

and if $vw \leq 0$ then,

$$\begin{aligned} \mathbb{P}\left(\max_{k:\sigma_k \in [r,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t,r} = v, \widehat{W}_{t,t} = w\right) \\ \leq C' \frac{(v^- + e^{-\frac{3}{2}v^+})(w^- + e^{-\frac{3}{2}w^+})}{t-r} \exp\left(\frac{(v-w)^2}{2(t-r)}\right). \end{aligned} \quad (3.10)$$

Also, there exists non-increasing functions $g : \mathbb{R} \rightarrow (0, \infty)$ and $f^{(r)} : \mathbb{R} \rightarrow (0, \infty)$ for $r \geq 0$, such that for all such r

$$\mathbb{P}\left(\max_{k:\sigma_k \in [r,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t,r} = v, \widehat{W}_{t,t} = w\right) \sim 2 \frac{f^{(r)}(v)g(w)}{t-r} \text{ as } t \rightarrow \infty, \quad (3.11)$$

uniformly in v, w satisfying $v, w < 1/\epsilon$ and $(v^- + 1)(w^- + 1) \leq t^{1-\epsilon}$ for any fixed $\epsilon > 0$. Moreover,

$$\lim_{v \rightarrow \infty} \frac{f^{(r)}(-v)}{v} = \lim_{w \rightarrow \infty} \frac{g(-w)}{w} = 1, \quad (3.12)$$

for any $r \geq 0$. Finally there exists $f : \mathbb{R} \rightarrow (0, \infty)$ such that for all $v \in \mathbb{R}$,

$$f^{(r)}(v) \xrightarrow{r \rightarrow \infty} f(v). \quad (3.13)$$

Proof. Given r, t, v, w satisfying the above assumptions, let $t^{(r)} := t - r$. By tilting and shifting we can replace $\widehat{W}_{t,s}$ everywhere inside the probability on the left hand side of (3.9) by $\widehat{W}_{t,s} + \gamma_{t,r} + \frac{s-r}{t^{(r)}}(\gamma_{t,t} - \gamma_{t,r}) = W_s - \gamma_{t,s} + \gamma_{t,r} + \frac{s-r}{t^{(r)}}(\gamma_{t,t} - \gamma_{t,r})$. Setting

$$\gamma_{t^{(r)},s}^{(r)} := \gamma_{t,s+r} - \gamma_{t,r} - \frac{s}{t^{(r)}}(\gamma_{t,t} - \gamma_{t,r}), \quad Y_s^{(r)} := -\widehat{h}_{s+r}^{(s+r)*}, \quad (3.14)$$

and using shift law invariance of W and \mathcal{N} , the left hand side of (3.9) now reads

$$\mathbb{P}\left(\max_{k:\sigma_k \in [0,t^{(r)}]} (W_{\sigma_k} - \gamma_{t^{(r)},\sigma_k}^{(r)} - Y_{\sigma_k}^{(r)}) \leq 0 \mid W_0 = v, W_{t^{(r)}} = w\right). \quad (3.15)$$

Next, we want to apply Propositions 2.1, 2.2 and 2.4. We just need to make sure that the conditions required by these propositions hold. By assumption, (2.2) holds with $\lambda = 2$ and thanks to Lemma 2.9 we know that (2.1) holds with any δ small enough uniformly in r . Finally, using Lemma 3.3 noting that the middle expression in (3.7) is exactly $\gamma_{t^{(r)},s}^{(r)}$, we have

$$-1 \leq \gamma_{t^{(r)},s}^{(r)} \leq 1 + \log^+ \wedge^{t^{(r)}}(s) : 0 \leq r \leq t, \text{ and } 0 \leq s \leq t^{(r)}, \quad (3.16)$$

which is precisely Condition (2.7) with $\delta = 1$. This implies that for any $r \geq 0$ both statements in Proposition 2.1 are in force, provided that δ is small enough. In particular, by decreasing δ if necessary, we may and will assume that $\sqrt{2\lambda}(1 - \delta) = 2(1 - \delta) \geq 3/2$, which yields (3.9) and (3.10).

Turning now to (3.11) and (3.13), a bit of algebra shows that for fixed r and all $s \geq 0$

$$\lim_{t^{(r)} \rightarrow \infty} \gamma_{t^{(r)},s}^{(r)} = \log^+(s+r) - \log^+ r =: \gamma_{\infty,s}^{(r)}, \quad \lim_{t^{(r)} \rightarrow \infty} \gamma_{t^{(r)},t^{(r)}-s}^{(r)} = 0 =: \gamma_{\infty,-s}^{(r)}, \quad (3.17)$$

while the convergence of the centered maximum gives, $Y_s^{(r)} \Rightarrow Y$ as $s \rightarrow \infty$ or $r \rightarrow \infty$, where Y has the limiting law of the centered maximum. Moreover, for all $s \geq 0$ clearly $\gamma_{\infty,s}^{(r)} \rightarrow 0$ as $r \rightarrow \infty$. Therefore the conditions of Proposition 2.2 and Proposition 2.4 are satisfied implying (3.11) and (3.13). \square

Next we have the following sharp entropic-repulsion result, which follows directly from Proposition 2.5.

Lemma 3.5. *For all $M \geq 1$,*

$$\sup_{s \geq 1} \limsup_{t \rightarrow \infty} \sqrt{s} \mathbb{P} \left(\max_{u \in [s, t-s]} \widehat{W}_{t,u} \geq -M \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0, \max_{\sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right) < \infty. \quad (3.18)$$

Proof. We have already verified in the proof of Proposition 3.4 that Conditions (2.1), (2.2) and (2.7), (2.8) hold with δ small enough and $\lambda = 2$. Therefore, it remains to check (2.14). To this end, observe that $\gamma_{t,u} - \gamma_{t,r}(u/r) = \gamma_{r,u}$, while $\gamma_{t,u'} - \gamma_{t,r}(t-u')/(t-r) = \gamma_{t-r,u}^{(r)}$ as defined in (3.14). Then (3.16) shows that condition (2.14) holds and we can then appeal to Proposition 2.5 with $x = y = 0$ to claim the result. \square

4 Truncated Moments of the Level Set Size

The goal in this section is to estimate the first and second moments of the number of particles lying above $m_t + v$ for $v \in \mathbb{R}$. Since the expectation of such quantities blows up as $t \rightarrow \infty$, one has to introduce a truncation event. Unlike the usual truncation event (introduced by Bramson in [12]), whereby the trajectory of such particle is constrained to lie below a curve, we choose to use the event that the global maximum stays below a certain value, namely $\{h_t^* \leq m_t + u\}$. This truncation can be more conveniently used later, when we derive cluster properties (Section 5). In light of the tightness of the centered global maximum, the probability of this event tends to 0 when $u \rightarrow \infty$ uniformly in t . Therefore, for the sake of distributional results, we can always work under this restriction and remove it just in the very end.

Recall the definition of the extremal process from (1.3). Since for every Borel set $A \subset \mathbb{R}$

$$\mathcal{E}_t(A) 1_{\{\widehat{h}_t^* \leq u\}} = \sum_{x \in L_t} 1_{\{\widehat{h}_t(x) \in A, \widehat{h}_t^* \leq u\}}, \quad \mathcal{E}_t(A)^2 1_{\{\widehat{h}_t^* \leq u\}} = \sum_{x,y \in L_t} 1_{\{\widehat{h}_t(x) \in A, \widehat{h}_t(y) \in A, \widehat{h}_t^* \leq u\}}. \quad (4.1)$$

we can use the spinal decomposition in the form of the many-to-one and many-to-two lemmas in Subsection 2.2, to compute the expectation of the quantities above, provided we can estimate the probabilities, under the corresponding spine measures, of the events in the sums, with x, y replaced by the spine particles $X_t(1), X_t(2)$, respectively. We start with the first moment.

4.1 First Moment

Recall that the one-spine measure as introduced in Subsection 2.2 is denoted by $\tilde{\mathbb{P}}$ and the corresponding expectation is $\tilde{\mathbb{E}}$.

Lemma 4.1. *There exists $C, C' > 0$ such that for all $t \geq 0$ and $v \leq u$,*

$$\tilde{\mathbb{P}}(\hat{h}_t(X_t) \geq v, \hat{h}_t^* \leq u) \leq Ce^{-t}e^{-\sqrt{2}v}(u-v+1)(u^++1)(e^{-\frac{v^2}{4t}} + e^{\frac{v}{2}}), \quad (4.2)$$

in addition, if $u \leq 0$ then we also have that

$$\tilde{\mathbb{P}}(\hat{h}_t(X_t) \geq v, \hat{h}_t^* \leq u) \leq C'e^{-t}e^{-\sqrt{2}v}(u-v+1)e^{-\frac{3}{2}u^-}. \quad (4.3)$$

Moreover with $g : \mathbb{R} \rightarrow (0, \infty)$ from Lemma 3.4 we have that uniformly in u, v satisfying $|u| \leq 1/\epsilon$ and $t^\epsilon < u-v < t^{1-\epsilon}$, for any fixed $\epsilon > 0$,

$$\tilde{\mathbb{P}}(\hat{h}_t(X_t) \geq v, \hat{h}_t^* \leq u) \sim e^{-t}e^{-\sqrt{2}v-\frac{v^2}{2t}}(u-v+1)\frac{g(-u)}{\sqrt{\pi}} \quad \text{as } t \rightarrow \infty. \quad (4.4)$$

Proof. Starting with the first upper bound, we write the left hand side of (4.2) as the integral

$$\int_{w=v}^u \tilde{\mathbb{P}}(\hat{h}_t^* \leq u \mid \hat{h}_t(X_t) = w) \tilde{\mathbb{P}}(\hat{h}_t(X_t) \in dw). \quad (4.5)$$

Using the second part of Lemma 3.1 and then the first upper bound in Lemma 3.4, the conditional probability in the integral is bounded above by $Ct^{-1}(u-w+1)(u^++1)$. At the same time, $\hat{h}_t(X_t)$ is Gaussian with mean $-m_t := -\sqrt{2}t + \frac{3}{2\sqrt{2}}\log^+ t$ and variance t . Therefore,

$$\frac{\tilde{\mathbb{P}}(\hat{h}_t(X_t) \in dw)}{dw} = \frac{te^{-t}e^{-\sqrt{2}w}}{\sqrt{2\pi}} \exp\left(-\frac{(w-\frac{3}{2\sqrt{2}}\log^+ t)^2}{2t}\right) \leq Cte^{-t} \exp\left(-\sqrt{2}w - \frac{w^2}{4t}\right) \quad (4.6)$$

Using these inequalities in (4.5) we may bound the integral by

$$Ce^{-t}(u^++1)(u-v+1) \times \begin{cases} e^{-\frac{v^2}{4t}-\sqrt{2}v} & : v \geq (-\sqrt{8}+\eta)t, \\ e^{(2+\eta)t} & : v < (-\sqrt{8}+\eta)t, \end{cases} \quad (4.7)$$

with any $\eta > 0$ and $C = C(\eta) > 0$. Choosing η small enough, the last factor in (4.7) can be bounded by $e^{-\sqrt{2}v}(e^{-v^2/4t} + e^{v/2})$, which gives the upper bound.

Now, if $u \leq 0$, then $v \leq w \leq 0$ and consequently the left hand side of (4.6) can be bounded by $Cte^{-t} \times \exp(-\sqrt{2}w - \frac{w^2}{2t})$. Observing that $w-u \leq 0$, we now use the second upper bound in Lemma 3.4 to estimate the first term in the integral in (4.5). The probability in question is now bounded by

$$Ce^{-t}e^{-\frac{3}{2}u^-} \int_{w=v}^u e^{-\sqrt{2}w}(u-w+1)dw, \quad (4.8)$$

which is smaller than the right hand side of (4.3) for a proper constant $C' > 0$.

As for the asymptotic statement, we use Lemma 3.1 again and then Lemma 3.4 with $r = 0$ for the first term in the integral, but this time we use (3.11) in order to obtain asymptotics. This gives

$$\tilde{\mathbb{P}}(\hat{h}_t^* \leq u \mid \hat{h}_t(X_t) = w) \sim 2 \frac{f^{(0)}(w - u)g(-u)}{t}, \quad (4.9)$$

as $t \rightarrow \infty$, uniformly in u, v as specified in the statement and any $w \in [v, u]$. Using (4.6) we also have that uniformly in $w \in [v, u]$

$$\frac{\hat{\mathbb{P}}(\hat{h}_t(X_t) \in dw)}{dw} \sim \frac{te^{-t}}{\sqrt{2\pi}} \exp\left(-\sqrt{2}w - \frac{w^2}{2t}\right) \quad \text{as } t \rightarrow \infty. \quad (4.10)$$

Plugging these estimates in (4.5), the integral there is uniformly asymptotic to

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} e^{-t} g(-u) \int_{w=v}^u f^{(0)}(w - u) \exp\left(-\sqrt{2}w - \frac{w^2}{2t}\right) dw \\ &= \sqrt{\frac{2}{\pi}} e^{-t} g(-u) (u - v + 1) e^{-\sqrt{2}v - \frac{v^2}{2t}} \int_{y=0}^{u-v} \frac{f^{(0)}(v - u + y)}{u - v + 1} \exp\left(-\sqrt{2}y - \frac{y^2}{2t} - \frac{yv}{t}\right) dy, \end{aligned} \quad (4.11)$$

where we have also substituted $y = w - v$ to obtain the second line above.

Since $f^{(0)}(-x) \sim x$ as $x \rightarrow \infty$ and $u - v \geq t^\epsilon$ the ratio in the integrand is bounded by above and tends to 1 as $t \rightarrow \infty$, with convergence uniform in $y = o(t^\epsilon)$. Moreover, since $|v|t^{-1} = o(1)$ uniformly as $t \rightarrow \infty$, the above integral restricted to $y > \log t$ vanishes as $t \rightarrow \infty$. On the other hand, when $y \in [0, \log t]$ the integrand converges uniformly to $e^{-\sqrt{2}y}$ as $t \rightarrow \infty$, implying that the integral itself converges uniformly to $\int_0^\infty e^{-\sqrt{2}y} dy = 1/\sqrt{2}$, which yields (4.4). \square

We are now in a position to estimate the first moment of $\mathcal{E}_t([v, \infty))$ under the restriction that $\hat{h}_t^* \leq u$.

Lemma 4.2. *There exists $C > 0$ such that for all $t \geq 0$ and $v \leq u$,*

$$\mathbb{E}\left(\mathcal{E}_t([v, \infty)); \hat{h}_t^* \leq u\right) \leq C e^{-\sqrt{2}v} (u - v + 1) (u^+ + 1) (e^{-v^2/4t} + e^{v/2}). \quad (4.12)$$

Moreover with $g : \mathbb{R} \rightarrow (0, \infty)$ from Lemma 3.4 we have that uniformly in u, v satisfying $|u| \leq 1/\epsilon$ and $t^\epsilon < u - v < t^{1-\epsilon}$, for any fixed $\epsilon > 0$,

$$\mathbb{E}\left(\mathcal{E}_t([v, \infty)); \hat{h}_t^* \leq u\right) \sim e^{-\sqrt{2}v - \frac{v^2}{2t}} (u - v + 1) \frac{g(-u)}{\sqrt{\pi}} \quad \text{as } t \rightarrow \infty. \quad (4.13)$$

Proof. Writing $\mathcal{E}_t([v, \infty)) 1_{\{\hat{h}_t^* \leq u\}}$ as $\sum_{x \in L_t} F(x)$ with $F(x) = 1_{\{\hat{h}_t(x) \geq v, \hat{h}_t^* \leq u\}}$, we may apply the (many-to-one) Lemma 2.6 and then use Lemma 4.1 to estimate the resulting integral, the result follows. \square

4.2 Second Moment

For the second moment we only need an upper bound. Recall that the two-spine measure as introduced in Subsection 2.2 is denoted by $\tilde{\mathbb{P}}^{(2)}$ and the corresponding expectation is $\tilde{\mathbb{E}}^{(2)}$.

Lemma 4.3. *There exists $C > 0$ such that for all $0 \leq r \leq t$ and $v \leq u$,*

$$\begin{aligned} & \tilde{\mathbb{P}}^{(2)}\left(\min\{\hat{h}_t(X_t(1)), \hat{h}_t(X_t(2))\} \geq v, \hat{h}_t^* \leq u \mid d(X_t(1), X_t(2)) = r\right) \\ & \leq C \frac{e^{-t-r}}{1 + (r \wedge (t-r))^{3/2}} e^{\sqrt{2}u} (u^+ + 1) e^{-2\sqrt{2}v} (u-v+1)^2 \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u-v)}{2}}\right). \end{aligned} \quad (4.14)$$

Proof. In light of Remark 2.8, by conditioning further on the position of $h_{t-r}(X_{t-r}(1))$, which is the position of $h_{t-r}(X_{t-r}(2))$ as well, the left hand side of (4.14) is equal to

$$\int_z \tilde{\mathbb{P}}\left(\hat{h}_{t-r}(X_{t-r}) - m_{t,r} \in dz, \hat{h}_t^*(B_r(X_t)^c) \leq u\right) \times \tilde{\mathbb{P}}\left(\hat{h}_r(X_r) \geq v-z, \hat{h}_r^* \leq u-z\right)^2, \quad (4.15)$$

where $m_{t,r} := m_t - m_r - m_{t-r} = \frac{3}{2\sqrt{2}}(\log^+ r + \log^+(t-r) - \log^+ t)$ and $X(t)$ is the one-spine particle. Before estimating each of the terms in the integrand separately, we first observe that $m_{t,r}$ above is always non-negative and satisfies

$$m_{t,r} - \frac{3}{2\sqrt{2}} \log^+ \wedge^t(r) \in [-\log 2, 0]. \quad (4.16)$$

To estimate the first term in the integrand, we use Lemma 3.1 to express it as

$$\tilde{\mathbb{P}}(\hat{h}_{t-r}(X_{t-r}) - m_{t,r} \in dz) \mathbb{P}\left(\max_{k: \sigma_k \in [r,t]} (\widehat{W}_{t,\sigma_k} + \hat{h}_{\sigma_k}^*) \leq 0 \mid \widehat{W}_{t,r} = z + m_{t,r} - u, \widehat{W}_{t,t} = -u\right). \quad (4.17)$$

In the first probability above, $\hat{h}_{t-r}(X_{t-r})$ has Gaussian distribution with mean $-m_{t-r}$ and variance $t-r$. Therefore its probability density function at $dz + m_{t,r}$ is explicitly given by

$$\frac{1}{\sqrt{2\pi(t-r)}} \exp\left(-\frac{(\sqrt{2}(t-r) - \frac{3}{2\sqrt{2}} \log^+(t-r) + m_{t,r} + z)^2}{2(t-r)}\right) \leq C(t-r) e^{-(t-r) - \sqrt{2}(z+m_{t,r})}, \quad (4.18)$$

where we have used the bound on $m_{t,r}$ and the fact that $z \frac{\log^+(t-r)}{t-r} - C' \frac{z^2}{t-r}$ is bounded uniformly in t, r and z for any $C' > 0$. Next, we use (3.9) to bound the conditional probability in (4.17) by $C(t-r)^{-1}(u^+ + 1)((u-z-m_{t,r})^+ + 1)$. So, it remains to estimate the squared term in the integrand of (4.15). If $z \leq u$ we use (4.2), to bound the squared term by

$$\begin{aligned} & C \left(e^{-r} e^{-\sqrt{2}(v-z)} (u-v+1)(u-z+1) (e^{-(v-z)^2/4t} + e^{(v-z)/2}) \right)^2 \\ & \leq C e^{-2r} (u-v+1)^2 e^{-2\sqrt{2}v} e^{2\sqrt{2}z} (u-z+1)^2 (e^{-(v-z)^2/4t} + e^{v-z}). \end{aligned} \quad (4.19)$$

Otherwise, if $z > u$, we use (4.2) for one factor and (4.3) for the other. This gives

$$\begin{aligned} C \left(e^{-r} e^{-\sqrt{2}(v-z)} (u-v+1) \left(e^{-(v-z)^2/4t} + e^{(v-z)/2} \right) \right) & \left(e^{-r} e^{-\sqrt{2}(v-z)} (u-v+1) e^{-\frac{3}{2}(z-u)} \right) \\ & = C e^{-2r} (u-v+1)^2 e^{-2\sqrt{2}v} e^{2\sqrt{2}z} e^{-\frac{3}{2}(z-u)} \left(e^{-(v-z)^2/4t} + e^{(v-z)/2} \right). \end{aligned} \quad (4.20)$$

We now split the integral in (4.15) according to whether $z \leq u$ or $z > u$. In the former range, we use (4.19) and bound it by

$$e^{-t-r-\sqrt{2}m_{t,r}}(u^++1)(u-v+1)^2 e^{-2\sqrt{2}v} \int_{z \leq u} \left(e^{-\frac{(v-z)^2}{4t} + \sqrt{2}z} + e^{v+(\sqrt{2}-1)z} \right) (u-z+1)^3 dz. \quad (4.21)$$

Distributing the $(u-z+1)^3$, the integral of the second term is bounded by $Ce^{\sqrt{2}u-(u-v)}$. For the integral of the first term, we observe that the exponent $-(v-z)^2/(4t) + \sqrt{2}z$ is maximized at $z = 2\sqrt{2}t + v$. Therefore, if $u < 2\sqrt{2}(1-\eta)t + v$ for some $\eta > 0$, the integral of the first term is bounded by a constant times the value of the integrand at u , which gives the bound $Ce^{\sqrt{2}u}e^{-(u-v)^2/4t}$, with $C > 0$ depending on η . On the other hand, if $u > 2\sqrt{2}(1-\eta)t + v$, then we integrate the first term in absolute value over all \mathbb{R} , thereby obtaining the upper bound

$$Cte^{2t+\sqrt{2}v}(u-v-2\sqrt{2}t+1)^3 \leq Ce^{\sqrt{2}u}e^{-(u-v)/2}, \quad (4.22)$$

for η small enough, where we have used that $2\sqrt{2}(1-\eta)t \leq u-v$. Putting all of these together, the integral in (4.21) can always be bounded by

$$Ce^{\sqrt{2}u} \left(e^{-(u-v)/2} + e^{-(u-v)^2/4t} + e^{-(u-v)} \right) \leq Ce^{\sqrt{2}u} \left(e^{-(u-v)^2/4t} + e^{-(u-v)/2} \right). \quad (4.23)$$

Returning to the integral in (4.15), in the range $z \geq u$ we use (4.20) to get the upper bound

$$e^{-t-r-\sqrt{2}m_{t,r}}(u^++1)(u-v+1)^2 e^{-2\sqrt{2}v} \int_{z \geq u} e^{\sqrt{2}z-\frac{3}{2}(z-u)} \left(e^{-(v-z)^2/4t} + e^{(v-z)/2} \right) dz. \quad (4.24)$$

The sum of the first two exponents maximizes at $z = v - (3-2\sqrt{2})t \leq u - (3-2\sqrt{2})t$, while the sum of the first and the last exponents always maximizes at u . This means that $z = u$ determines the bound on the integral and gives $Ce^{\sqrt{2}u} \left(e^{-(u-v)^2/4t} + e^{-(u-v)/2} \right)$ as an upper bound exactly as in the previous range.

Altogether, the integral in (4.15) is bounded above by

$$Ce^{-t-r-\sqrt{2}m_{t,r}}(u^++1)(u-v+1)^2 e^{-2\sqrt{2}v} e^{\sqrt{2}u} \left(e^{-(v-u)^2/4t} + e^{-(u-v)/2} \right). \quad (4.25)$$

To make the identification with the right hand side of (4.14) just notice that (4.16) implies

$$e^{-\sqrt{2}m_{t,r}} \leq C(1+(r \wedge (t-r))^{-3/2}), \quad (4.26)$$

proving the statement. \square

We can now use the many-to-two lemma to bound the second moment.

Lemma 4.4. *There exists $C > 0$ such that for all $v \leq u$,*

$$\mathbb{E}\left(\mathcal{E}_t([v, \infty))^2; \hat{h}_t^* \leq u\right) \leq e^{-2\sqrt{2}v}(u-v+1)^2 e^{\sqrt{2}u}(u^++1)(e^{-(u-v)^2/4t} + e^{-(u-v)/2}). \quad (4.27)$$

Proof. In light of the second equation in (4.1) we can use (the many-to-two) Lemma 2.7 with $F(x, y) = 1_{\{\min\{\hat{h}_t(y), \hat{h}_t(x)\} \geq v, \hat{h}_t^* \leq u\}}$, thereby obtaining

$$e^{3t}\widetilde{\mathbb{E}}^{(2)}\left(e^{-d(X_t(1), X_t(2))}; \min\{\hat{h}_t(X_t(1)), \hat{h}_t(X_t(2))\} \geq v, \hat{h}_t^* \leq u\right). \quad (4.28)$$

Conditioning on $d(X_t(1), X_t(2))$ and recalling that the distribution of $t - d(X_t(1), X_t(2))$ is exponential with rate 2 truncated at t (see Remark 2.8), we may use Lemma 4.3 to bound the last display by

$$\begin{aligned} & C e^{\sqrt{2}u}(u^++1) e^{-2\sqrt{2}v}(u-v+1)^2 (e^{-(u-v)^2/4t} + e^{-(u-v)/2}) \\ & \times e^{3t} \left(e^{-3t} + \int_{r=0}^t \frac{e^{-t-r}}{1+(r \wedge (t-r))^{3/2}} e^{-r} e^{-2(t-r)} dr \right). \end{aligned} \quad (4.29)$$

Since the term on the second line is bounded by a constant, the result follows. \square

5 Proofs of Cluster Level Set Propositions

The aim in this section is to prove the cluster properties stated in Subsection 1.2.2. We start with the following lemma that characterizes the limiting cluster distribution in terms of the cluster around the spine particle, conditioned to be the global maximum. Recall the spinal decomposition from Subsection 2.2 and that in particular X_t denotes the spine particle at time t .

Lemma 5.1. *Let $\mathcal{C} \sim \nu$ be distributed according to the cluster law. Then for any Borel set $B \subseteq \mathbb{M}$ and any $u \in \mathbb{R}$,*

$$\mathbb{P}(\mathcal{C} \in B) = \lim_{t \rightarrow \infty} \widetilde{\mathbb{P}}(\mathcal{C}_{t, r_t}(X_t) \in B \mid \hat{h}_t(X_t) = \hat{h}_t^* = u), \quad (5.1)$$

where $\mathcal{C}_{t, r_t}(X_t) := \sum_{y \in B_{t, r_t}(X_t)} \delta_{h_t(y) - h_t(X_t)}$ denotes the cluster around X_t as defined in (1.7).

Proof. The proof of Theorem 2.3 in [2] shows that $\mathbb{P}(\mathcal{C}_{t, r_t}(X_t^*) \in B) \rightarrow \mathbb{P}(\mathcal{C} \in B)$ as $t \rightarrow \infty$, where $\mathcal{C}_{t, r_t}(X_t^*)$ is the cluster around the highest particle $X_t^* := \operatorname{argmax}_{x \in L_t} h_t(x)$. Thanks to the product structure of the intensity measure governing the limiting Poisson point process and the absolute continuity of its first coordinate, the above limit still holds if we condition on $h_t^* = m_t + u$ for any $u \in \mathbb{R}$, namely

$$\mathbb{P}(\mathcal{C} \in B) = \lim_{t \rightarrow \infty} \mathbb{P}(\mathcal{C}_{t, r_t}(X_t^*) \in B \mid h_t^* = m_t + u). \quad (5.2)$$

We rewrite the probabilities inside the limit as $\mathbb{E}(\sum_{x \in L_t} 1_{\{\mathcal{C}_{t,r_t}(x) \in B\} \cap \{x = X_t^*\}} \mid h_t^* = m_t + u)$ and use (the many-to-one) Lemma 2.6 twice, with $x \mapsto 1_{\{\mathcal{C}_{t,r_t}(x) \in B\} \cap \{\hat{h}_t^* = \hat{h}_t(x) \in du\}}$ and then with $x \mapsto 1_{\{\hat{h}_t^* = \hat{h}_t(x) \in du\}}$ as the random function $F(x)$, to obtain

$$\mathbb{P}(\mathcal{C}_{t,r_t}(X_t^*) \in B \mid h_t^* = m_t + u) = \frac{\tilde{\mathbb{P}}(\mathcal{C}_{t,r_t}(X_t) \in B, \hat{h}_t(X_t) = \hat{h}_t^* \in du)}{\tilde{\mathbb{P}}(\hat{h}_t(X_t) = \hat{h}_t^* \in du)}, \quad (5.3)$$

which is equal to the right hand side of (5.1). \square

For what follows in the section, we will mostly work with variants of the conditional probability $\tilde{\mathbb{P}}(\cdot \mid \hat{h}_t(X_t) = \hat{h}_t^*)$, in which case the configuration $\mathcal{C}_{t,r_t}(X_t)$ around the spine is exactly the configuration around the maximal particle X_t^* therefore we shorten the notation $\mathcal{C}_{t,r_t}(X_t)$ into

$$\mathcal{C}_{t,r_t}^* := \mathcal{C}_{t,r_t}(X_t) = \sum_{y \in B_{t,r_t}(X_t)} \delta_{h_t(y) - h(X_t)} \quad (5.4)$$

We can now begin proving the propositions in Subsection 1.2.2. We dedicate a subsection to each of these proofs.

5.1 Proof of Proposition 1.6

The proof of Proposition 1.6 follows readily from the two results below, whose proofs we postpone to the end of the section. The first one gives the $v \rightarrow \infty$ asymptotic of $\tilde{\mathbb{E}}\mathcal{C}_{t,r_t}^*([-v, 0])$.

Lemma 5.2. *There exists $C > 0$ such that as $t \rightarrow \infty$ and then $v \rightarrow \infty$,*

$$\tilde{\mathbb{E}}\left(\mathcal{C}_{t,r_t}^*([-v, 0]) \mid \hat{h}_t^* = \hat{h}_t(X_t) = 0\right) \sim Ce^{\sqrt{2}v}. \quad (5.5)$$

Whereas the second one provides upper bounds for the second moment of $\mathcal{C}_{t,r_t}^*([-v, 0])$.

Lemma 5.3. *There exists $C > 0$ such that for all $v \geq 0$,*

$$\limsup_{t \rightarrow \infty} \tilde{\mathbb{E}}\left((\mathcal{C}_{t,r_t}^*([-v, 0]))^2 \mid \hat{h}_t^* = \hat{h}_t(X_t) = 0\right) \leq C(v+1)e^{2\sqrt{2}v}. \quad (5.6)$$

Proof of Proposition 1.6. By Lemma 5.3, for all $v \geq 0$ there exist $t_0 \geq 0$ such that the collection of random variables $\{\mathcal{C}_{t,r_t}^*([-v, 0]) : t \geq t_0\}$ is uniformly integrable under the conditional measure $\tilde{\mathbb{P}}(\cdot \mid \hat{h}_t^* = \hat{h}_t(X_t) = 0)$ and therefore in light Lemma 5.1 with $u = 0$, the expectation of $\mathcal{C}_{t,r_t}^*([-v, 0])$ under this measure converges as $t \rightarrow \infty$ to the expectation of $\mathcal{C}([-v, 0])$ under ν , provided that \mathcal{C} does not charge $-v$ with positive probability. Although this is indeed the case, instead of proving this, given $v \in \mathbb{R}$, we can always find $v - 1/v < v' \leq v \leq v'' \leq v + 1/v$ such that v', v'' are not charged by \mathcal{C} with probability 1. The existence of such points is assured by the fact that the set of points which are charged with positive probability by \mathcal{C} is at most countable. Then by monotonicity,

$$\lim_{t \rightarrow \infty} \tilde{\mathbb{E}}\mathcal{C}_{t,r_t}^*([-v', 0]) = \mathbb{E}\mathcal{C}([-v', 0]) \leq \mathbb{E}\mathcal{C}([-v, 0]) \leq \mathbb{E}\mathcal{C}([-v'', 0]) = \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}\mathcal{C}_{t,r_t}^*([-v'', 0]). \quad (5.7)$$

Now, the first and last quantities are asymptotically equivalent to $Ce^{\sqrt{2}v'}$ and $Ce^{\sqrt{2}v''}$ respectively, which in light of the choice of v', v'' are also asymptotic to $Ce^{\sqrt{2}v}$. This shows the first part of the proposition with $C_* = C$, where C is the constant in Lemma 5.2.

The second part of the proposition follows from Lemma 5.1, Lemma 5.3 and an application of Fatou's lemma, whenever $[-v, 0]$ is a stochastic continuity sets under \mathcal{C} (as a process on \mathbb{R}_-). As before, if this is not the case, we pick v'' as before and use monotonicity again. \square

It remains therefore to prove Lemma 5.2 and Lemma 5.3 and at this point we can appeal to Lemma 3.2 to represent the cluster \mathcal{C}_{t,r_t}^* in terms of the decorated random walk process of Section 3. This is the content of the next lemma, but before we can state it, we need several new definitions and/or abbreviations. First, recall the random objects: W , H and \mathcal{N} from Section 3 and that \mathcal{E}_t^s is the extremal process of h_t^s . Next, let us abbreviate for $t \geq 0$,

$$\mathcal{A}_t := \left\{ \max_{k: \sigma_k \in [0, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \right\}, \quad \widehat{\mathbb{P}}_t(\cdot) = \mathbb{P}(\cdot \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0), \quad (5.8)$$

with $\widehat{\mathbb{E}}_t$ the corresponding expectation. Finally, for $v \geq 0$ and $0 \leq s \leq t$ we set $j_{t,v}(s) := \widehat{E}_t J_{t,v}(s)$ where

$$J_{t,v}(s) := \mathcal{E}_s^s([-v, 0] - \widehat{W}_{t,s}) 1_{\{\widehat{h}_s^{s*} \leq -\widehat{W}_{t,s}\}} \times 1_{\mathcal{A}_t}, \quad (5.9)$$

and for $0 \leq s \leq s' \leq t$, also $j_{t,v}(s, s') := \widehat{E}_t J_{t,v}(s, s')$ where

$$J_{t,v}(s, s') := \mathcal{E}_s^s([-v, 0] - \widehat{W}_{t,s}) 1_{\{\widehat{h}_s^{s*} \leq -\widehat{W}_{t,s}\}} \times \mathcal{E}_{s'}^{s'}([-v, 0] - \widehat{W}_{t,s'}) 1_{\{\widehat{h}_{s'}^{s'*} \leq -\widehat{W}_{t,s'}\}} \times 1_{\mathcal{A}_t}. \quad (5.10)$$

We now have,

Lemma 5.4. *Let $v \geq 0$. Then.*

$$\widetilde{\mathbb{E}}\left(\mathcal{C}_{t,r_t}^*([-v, 0]); \widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0\right) = 2 \int_{s=0}^{r_t} j_{t,v}(s) ds. \quad (5.11)$$

and

$$\widetilde{\mathbb{E}}\left((\mathcal{C}_{t,r_t}^*([-v, 0]))^2; \widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0\right) = 4 \int_{s,s'=0}^{r_t} j_{t,v}(s, s') ds ds' + 2 \int_{s=0}^{r_t} j_{t,v}(s, s) ds. \quad (5.12)$$

Proof. Let us start with (5.11). By Lemma 3.1 with $r = 0$, $u = w = 0$ and Lemma 3.2 with $r = r_t$ (ignoring the law of $h_{t-s}(X_{t-s})$), we may write the left hand side as

$$\widehat{E}_t \int_{s=0}^{r_t} J_{t,v}(s) \mathcal{N}(ds). \quad (5.13)$$

Since \mathcal{N} is a Poisson point process on \mathbb{R}_+ with intensity $2dx$, its associated Palm kernel can be written as $(\mathbb{P}(\mathcal{N}_s \in \cdot) : s \geq 0)$ where $(\mathcal{N}_s : s \geq 0)$ is a family of point processes such that $\mathcal{N}_s \stackrel{\text{law}}{=} \mathcal{N} + \delta_s$, assumed to be defined alongside W and H and independent of them. Now,

conditional on $\mathcal{F} := \sigma(W, H)$ the random function $J_{t,v}$ depends only on \mathcal{N} (through the last indicator in its definition). Therefore by Palm-Campbell theorem (see, e.g. Proposition 13.1.IV in [15]) and independence between \mathcal{N}_s and \mathcal{F} ,

$$\widehat{E}_t\left(\int_{s=0}^{r_t} J_{t,v}(s)\mathcal{N}(ds) \mid \mathcal{F}\right) = 2 \int_{s=0}^{r_t} \widehat{E}_t(J_{t,v}(\mathcal{N}_s, s) \mid \mathcal{F}) ds, \quad \widehat{P}_t - \text{a.s.}, \quad (5.14)$$

where $J_{t,v}(\mathcal{N}_s, s)$ is defined as in (5.9) only with \mathcal{N}_s replacing \mathcal{N} . However, because of the middle indicator in definition (5.9), there is in fact no difference between $J_{t,v}(\mathcal{N}_s, s)$ and $J_{t,v}(s)$. Taking now expectation with respect to \widehat{E}_t and using Fubini's theorem to exchange between the integral and the expectation on the right hand side, we obtain (5.11).

The second of the lemma is quite similar. We first write the left hand side of (5.12) as

$$\widehat{E}_t\left(\int_{s=0}^{r_t} J_{t,v}(s)\mathcal{N}(ds)\right)^2 = \widehat{E}_t \int_{s,s'=0}^{r_t} J_{t,v}(s, s')\mathcal{N}^2(ds \times ds'), \quad (5.15)$$

where \mathcal{N}^2 is the product measure of \mathcal{N} with itself. Letting $(\mathcal{N}_{s,s'} : s, s' \geq 0)$ be a collection of point process which are independent of W and H and with $\mathcal{N}_{s,s'} \stackrel{\text{law}}{=} \mathcal{N} + \delta_s + 1_{s' \neq s} \delta_{s'}$, we now use the second order Palm-Campbell Theorem (see, e.g. Ex 13.1.11 in [15] or alternatively just apply the usual theorem to \mathcal{N}^2). This shows that the last expectation is equal to

$$\int_{s,s'=0}^{r_t} \widehat{E}_t(J_{t,v}(\mathcal{N}_{s,s'}, s, s'))\mathcal{M}(ds \times ds') = \int_{s,s'=0}^{r_t} \widehat{E}_t(J_{t,v}(s, s'))\mathcal{M}(ds \times ds'), \quad (5.16)$$

where in the first integral $J_{t,v}(\mathcal{N}_{s,s'}, s, s')$ is defined as $J_{t,v}(s, s')$ only with $\mathcal{N}_{s,s'}$ replacing \mathcal{N} , again making no difference, and in the second integral \mathcal{M}^2 is the intensity measure of the process \mathcal{N}^2 on \mathbb{R}_+^2 . Since \mathcal{M}^2 satisfies $\mathcal{M}^2(A) := 4 \int_{s,s'=0}^{\infty} 1_A(s, s') ds ds' + 2 \int_{s=0}^{\infty} 1_A(s, s) ds$ for all Borel sets $A \subseteq \mathbb{R}_+^2$, the result follows. \square

Next, we need asymptotics and bounds on $j_{t,v}(s)$ and $j_{t,v}(s, s')$. This is where the results of Section 4 will be used. For what comes next, given $M \geq 0$, we shall need the following refinements of $J_{t,v}(s)$ from (5.9):

$$J_{t,v}^{<M}(s) = J_{t,v}(s) 1_{\{|\widehat{W}_{t,s}| < M\}}, \quad J_{t,v}^{\geq M}(s) = J_{t,v}(s) 1_{\{|\widehat{W}_{t,s}| \geq M\}}, \quad (5.17)$$

with $j_{t,v}^{<M}(s)$, $j_{t,v}^{\geq M}(s)$ the respective expectations under $\widehat{\mathbb{E}}_t$. We start with upper bounds.

Lemma 5.5. *There exists $C, C' > 0$ such that for all $t \geq 0$, $0 \leq s \leq t/2$, $v \geq 0$ and $M \geq 0$,*

$$j_{t,v}^{\geq M}(s) \leq C \frac{e^{\sqrt{2}v}(v+1)}{t(s+1)\sqrt{s}} \times e^{-C'M} \left(e^{-\frac{v^2}{16s}} + e^{-\frac{v}{2}} \right). \quad (5.18)$$

Also, there exists $C > 0$ such that for all $t \geq 0$, $0 \leq s \leq s' \leq t/2$ and $v \geq 0$,

$$j_{t,v}(s, s') \leq C \frac{(v+1)^2 e^{2\sqrt{2}v} \left(e^{-\frac{v^2}{16s}} + e^{-\frac{v}{4}} \right) \left(e^{-\frac{v^2}{16s'}} + e^{-\frac{v}{4}} \right)}{t(s+1)(s'-s+1)\sqrt{s}(s'-s+1_{s=s'})} \quad (5.19)$$

Proof. Starting with the first inequality, by conditioning on $\widehat{W}_{t,s}$ we write $j_{t,v}^{\geq M}(s)$

$$\int_{|z| \geq M} q_t((0,0);(s,z)) \times e_{s,v}(z) \times q_t((s,z);(t,0)) \times p_t(s,z) dz, \quad (5.20)$$

where:

$$q_t((s_1, z_1), (s_2, z_2)) := \mathbb{P}\left(\max_{k: \sigma_k \in [s_1, s_2]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{*}) \leq 0 \mid \widehat{W}_{t, s_1} = z_1, \widehat{W}_{t, s_2} = z_2\right), \quad (5.21)$$

$$e_{s,v}(z) := \mathbb{E}(\mathcal{E}_s([-v, 0] - z); \widehat{h}_s^* \leq -z), \quad (5.22)$$

$$p_t(s, z) := \mathbb{P}(\widehat{W}_{t,s} \in dz \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0) / dz. \quad (5.23)$$

Observe that the definition of q_t above does not change if we replace $\widehat{W}_{t,u}$ by $\widehat{W}_{t',u}$ for any $t' \geq s_2$ everywhere inside the probability brackets. Indeed, recalling the definition of $\widehat{W}_{t,u}$ in (3.1), we see that the difference $\widehat{W}_{t,u} - \widehat{W}_{t',u}$ is a (deterministic) linear function of u , which is lost under the conditioning, because of the Gaussian law of \widehat{W} . In particular, we can rewrite the integral as

$$\int_{|z| \geq M} q_s((0,0);(s,z)) \times e_{s,v}(z) \times q_t((s,z);(t,0)) \times p_t(s,z) dz. \quad (5.24)$$

Now, conditioned on $\widehat{W}_{t,0} = \widehat{W}_{t,t}$ the law of $\widehat{W}_{t,s}$ is Gaussian with mean $-\gamma_{t,s}$ and variance $s(t-s)/t$. Thanks to the assumption $s \leq t/2$, the above variance always lies inside $[s/2, s]$ and hence $p_t(s, z)$ is smaller than $C s^{-1/2} e^{-(z+\gamma_{t,s})^2/2s}$. Using Lemma 3.4, either the first upper bound if $z \leq 0$ or the second if $z \geq 0$, we have

$$q_s((0,0);(s,z)) \times p_t(s,z) \leq C \frac{(z^- + e^{-\frac{3}{2}z^+})}{(s+1)\sqrt{s}}. \quad (5.25)$$

Above we have replaced s^{-1} from (3.9) by $(s+1)^{-1}$. To justify such replacement we notice that if $s \geq 1$, we can compensate for this change increasing the constant C . Whereas, if $s \in [0, 1]$ we just bound the left hand side above by $p_t(s, z)$ which is always smaller than the right hand side, again increasing the constant if necessary.

Using the upper bound in Lemma 4.2 to estimate $e_{s,v}(z)$ and again the first upper bound in Lemma 3.4 for $q_t((s,z);(t,0))$, the integral in (5.24) is always smaller than

$$\begin{aligned} & C \int_{|z| \geq M} \frac{(z^- + e^{-\frac{3}{2}z^+})}{(s+1)\sqrt{s}} \times e^{\sqrt{2}(v+z)}(v+1)(z^-+1) \left(e^{-\frac{(v+z)^2}{4s}} + e^{-\frac{v+z}{2}} \right) \times \frac{z^-+1}{t-s} dz \\ & \leq C \frac{e^{\sqrt{2}v}(v+1)}{t(s+1)\sqrt{s}} \int_{|z| \geq M} (z^-+1)^2 \left(z^- + e^{-\frac{3}{2}z^+} \right) \left(e^{-\frac{(v+z)^2}{4s} + \sqrt{2}z} + e^{-\frac{v+z}{2} + \sqrt{2}z} \right) dz. \end{aligned} \quad (5.26)$$

We now distribute the last parenthesis in the integrand and obtain two distinct integrals. Observing that $1/2 \leq \sqrt{2} \leq 3/2$, the first integral can be bounded above by $C e^{-C'M} e^{-v^2/16s}$ if

$z \geq -v/2$ and otherwise by $e^{-\sqrt{2}((v/2) \vee M)}((v/2) \vee M + 1)^3 \leq Ce^{-C'M}e^{-v/2}$. The second can just be bounded by $Ce^{-C'M}e^{-v/2}$. Combining these bounds the integral can always be bounded by $Ce^{-C'M}(e^{-v^2/16s} + e^{-v/2})$, which shows the first part of the lemma.

Moving on to the second, assume first that $s \neq s'$ and condition this time on $\widehat{W}_{t,s}$ and $\widehat{W}_{t,s'}$ to write $j_{t,v}(s, s')$ as

$$\int_{z,z'} q_s((0,0);(s,z)) \times e_{s,v}(z) \times q_{s'}((s,z);(s',z')) \times e_{s',v}(z') \times q_t((s',z');(t,0)) \times p_t((s,z);(s',z')) dz dz', \quad (5.27)$$

where $p_t((s,z);(s',z')) = \mathbb{P}(\widehat{W}_{t,s} \in dz, \widehat{W}_{t,s'} \in dz' \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0)$ and $e_{\cdot,v}(\cdot)$, $q_{\cdot}(\cdot)$ are defined as before. Then $p_t((s,z);(s',z'))$ satisfies

$$p_t((s,z);(s',z')) \leq \pi^{-1} \left(\frac{t}{s(s'-s)(t-s')} \right)^{1/2} \exp \left(-\frac{(z+\gamma_{t,s})^2}{2s} - \frac{(z'+\gamma_{t,s'})^2}{2(t-s')} \right), \quad (5.28)$$

As in the bound for $j_{t,v}(s)$, we now use the upper bounds in Lemma 3.4 for both q_s and q_t in the integrand, with the “right” bound chosen depending on whether z (respectively z') are positive or negative. This bounds $q_s((0,0);(s,z)) \times q_t((s',z');(t,0)) \times p_t((s,z);(s',z'))$ by

$$Ct^{-1}s^{-1/2}(s+1)^{-1}(s'-s)^{-1/2}(z^- + e^{-\frac{3}{2}z^+})(z'^- + e^{-\frac{3}{2}z'^+}), \quad (5.29)$$

where we have used that $t-s' \in [t/2, t]$ and again replaced the q_s term by 1 if $s \in [0, 1]$.

Using now Lemma 4.2 to bound the “ e -terms” and again the first upper bound in Lemma 3.4 for the remaining “ q -term” if $(s'-s) \geq 1$ or otherwise the trivial bound 1, the double integral in (5.27) is bounded up to a multiplicative factor by

$$\frac{e^{2\sqrt{2}v}(v+1)^2}{t(s+1)(s'-s+1)\sqrt{s(s'-s)}} \int_{z,z'} \left(z^- + e^{-\frac{3}{2}z^+} \right) \left(z'^- + e^{-\frac{3}{2}z'^+} \right) (z^- + 1)^2 (z'^- + 1)^2 \times e^{\sqrt{2}(z+z')} \left(e^{-\frac{(v+z)^2}{4s}} + e^{-\frac{(v+z)}{2}} \right) \left(e^{-\frac{(v+z')^2}{4s'}} + e^{-\frac{(v+z')}{2}} \right) dz dz'. \quad (5.30)$$

The above integral factors into two identical single variable integrals which are again equal to the integral in (5.26) when $M = 0$. Therefore the bound obtained there applies making the double integral smaller than $(e^{-v^2/(16s)} + e^{-v/2})(e^{-v^2/(16s')} + e^{-v/2})$ and whole last display smaller than the right hand side of (5.19).

Lastly we handle the case $s = s'$ and it is here where we need the second moment bound from Section 4. Again, we write $j_{t,v}(s, s)$ as

$$\int_z q_s((0,0);(s,z)) \times e_{s,v}^{(2)}(z) \times q_t((s,z);(t,0)) \times p_t(s,z) dz, \quad (5.31)$$

where $e_{s,v}^{(2)}(z) := \mathbb{E}(\mathcal{E}([-v+z], -z])^2; \hat{h}_s^* \leq -z)$. We now repeat the argument in the proof of (5.18) with $M = 0$, only that we use the bound on $e_{s,v}^{(2)}(z)$ from Lemma 4.4 instead of the bound on $e_{s,v}(z)$. This gives as an upper bound on $j_{t,v}(s, s)$,

$$C \frac{1}{t\sqrt{s}(s+1)} (v+1)^2 e^{2\sqrt{2}v} \left(e^{-\frac{v^2}{4s}} + e^{-\frac{v}{2}} \right) \int_z (z^- + 1)^2 (z^- + e^{-\frac{3}{2}z^+}) e^{\sqrt{2}z} dz. \quad (5.32)$$

The last integral is bounded by a constant and thus the whole expression can be made smaller than the right hand side of (5.19) if we properly tune the preceding constants. \square

Next, we need also asymptotics for $j_{t,v}^{<M}(s)$. This is given in the next lemma

Lemma 5.6. *There exists $C > 0$ such that as $t \rightarrow \infty$ followed by $v \rightarrow \infty$ and then $M \rightarrow \infty$,*

$$j_{t,v}^{<M}(s) \sim C t^{-1} s^{-3/2} v e^{\sqrt{2}v - \frac{v^2}{2s}}, \quad (5.33)$$

uniformly in $s \in [\eta v^2, v^2/\eta]$ for any fixed $\eta > 0$.

Proof. As in the previous lemma, we start by writing $j_{t,v}^{<M}(s)$ as the integral

$$j_{t,v}^{<M}(s) = \int_{|z| < M} q_s((0, 0); (s, z)) \times e_{s,v}(z) \times q_t((s, z); (t, 0)) \times p_t(s, z) dz, \quad (5.34)$$

with $q_s(\cdot), e_{s,v}(\cdot)$ and $p_t(\cdot)$ defined as before. We now use the corresponding asymptotic results, in place of the upper bounds we have used before, to derive asymptotics for the above integral when the limits are taken in the prescribed order.

Accordingly, let us first fix η, s, v and M and take $t \rightarrow \infty$. Conditioned on $\widehat{W}_{t,0} = \widehat{W}_{t,t} = 0$, the law of $\widehat{W}_{t,s}$ is Gaussian with mean $s(\log^+ t)/t - \log^+ s$ and variance $s(t-s)/t$. Hence, for all z and s fixed the density $p_t(s, z)$ of $\widehat{W}_{t,s}$ tends to

$$(2\pi s)^{-1/2} \exp\left(-\frac{(z - \log^+ s)^2}{2s}\right) \quad \text{as } t \rightarrow \infty, \quad (5.35)$$

and is bounded by $Cs^{-1/2}$ for all $t \geq s/2$ and any $z \in \mathbb{R}$ fixed. At the same time, by the third part of Lemma 3.4, we know that $q_t((s, z); (t, 0))$ is asymptotic equivalent to $2t^{-1}f^{(s)}(z)g(0)$ as $t \rightarrow \infty$. The first upper bound in the same lemma also says that $q_t((s, z); (t, 0))$ is smaller than $C(t-s)^{-1}(z^- + 1) < 2Ct^{-1}(z^- + 1)$ if $t \geq s/2$, which yields $f^{(s)}(z) \leq C(z^- + 1)$ for all $s > 0, z \in \mathbb{R}$ and t sufficiently large. Then, using the dominated convergence theorem, we can replace the quantities in the integrand of (5.34) with their asymptotic equivalences and obtain that the integral itself is asymptotic to

$$2 \frac{g(0)}{t\sqrt{2\pi s}} \int_{|z| < M} q_s((0, 0); (s, z)) e_{s,v}(z) f^{(s)}(z) \exp\left(-\frac{(z + \log^+ s)^2}{2s}\right) dz, \quad (5.36)$$

when $t \rightarrow \infty$ for fixed s and v .

Next, we keep M fixed and take $v \rightarrow \infty$. We will consider $s \in [\eta v^2, \eta^{-1} v^2]$, so that $s \rightarrow \infty$ as well. Then, by the third part of Lemma 3.4 again, we have that for any fixed z

$$q_s((0, 0); (s, z)) \sim 2 \frac{f^{(0)}(0)g(z)}{s} \quad \text{as } s \rightarrow \infty, \quad (5.37)$$

with $f^{(0)}(0), g(z) > 0$ from the lemma. Moreover the upper bounds in the same lemma also show that the left hand side above is smaller than $Cs^{-1}(z^- + e^{-3z^+/2})$ for all z and s . Again, this implies that $g(z) \leq C(z^- + e^{-3z^+/2})$ for all z with the constant independent of s . As for $f^{(s)}(z)$, the last part of Lemma 3.4 says that $f^{(s)}(z)$ is positive and it tends to $f(z) > 0$ as $s \rightarrow \infty$ and since we have established that $f^{(s)}(z) \leq C(z^- + 1)$ the same bound applies to the function f . Finally, we estimate $e_{s,v}(z)$ using Lemma 4.2 with u, v, t there replaced by $-z, -(v+z)$ and s respectively. Since $|z| \leq M$ and $\eta\sqrt{s} \leq v \leq \eta^{-1}\sqrt{s}$, the conditions of the lemma are satisfied with $\epsilon = 1/M$ and all s large enough, which yields

$$e_{s,v}(z) \sim v \frac{g(z)}{\sqrt{\pi}} \exp\left(\sqrt{2}(v+z) - \frac{(v+z)^2}{2s}\right) \sim \left(ve^{\sqrt{2}v - \frac{v^2}{2s}}\right) \frac{g(z)}{\sqrt{\pi}} e^{\sqrt{2}z}, \quad (5.38)$$

when $v \rightarrow \infty$ uniformly in $s \in [\eta v^2, \eta^{-1} v^2]$ and $|z| < M$. Combining all the above and using the dominated convergence theorem again, we see that the integral in (5.36) is asymptotic to

$$C \frac{ve^{\sqrt{2}v - \frac{v^2}{2s}}}{s} \int_{|z| < M} e^{\sqrt{2}z} f(z) g(z)^2 dz, \quad (5.39)$$

as $v \rightarrow \infty$ uniformly in s as required and for fixed M . Finally, in light of the positivity and upper bounds for f and g the last integral converges when $M \rightarrow \infty$ to a positive and finite constant. Collecting all the results together, we finish the proof. \square

We can now prove Lemma 5.2 and Lemma 5.3 and thereby complete the proof of Proposition 1.6.

Proof of Lemma 5.2. Fix first $v \geq 0$ and write $\tilde{\mathbb{E}}(\mathcal{C}_{t,r_t}^*([-v, 0]) \mid \hat{h}_t^* = \hat{h}_t(X_t) = 0)$ as

$$\frac{\tilde{\mathbb{E}}\left(\mathcal{C}_{t,r_t}^*([-v, 0]); \hat{h}_t^* \leq 0 \mid \hat{h}_t(X_t) = 0\right)}{\mathbb{P}(\hat{h}_t^* \leq 0 \mid \hat{h}_t(X_t) = 0)}. \quad (5.40)$$

An application of Lemma 3.1 with $r = u = w = 0$ followed by the third part of Lemma 3.4 shows that the denominator is asymptotic to Ct^{-1} as $t \rightarrow \infty$ with $C \in (0, \infty)$. Hence, it remains to treat the numerator.

Now let $M, \eta > 0$, assume t is large enough and use Lemma 5.4 to write the numerator as

$$2 \int_{s=\eta v^2}^{\eta^{-1} v^2} j_{t,v}^{<M}(s) ds + 2 \int_{s=0}^{r_t} \left(j_{t,v}(s) 1_{[\eta v^2, \eta^{-1} v^2]^c} + j_{t,v}^{\geq M}(s) 1_{[\eta v^2, \eta^{-1} v^2]} \right) ds. \quad (5.41)$$

We first want to claim that the second integral becomes negligible when $M \rightarrow \infty$ and $\eta \rightarrow 0$, in the asymptotic regime we consider. To this end, we observe that $j_{t,v}(s) = j_{t,v}^{\geq 0}(s)$, so the first upper bound in Lemma 5.5 may be used to estimate $j_{t,v}^{\geq M}(s)$ as well as $j_{t,v}(s)$ and bound the second integral above by $Ct^{-1}e^{\sqrt{2}v}(v+1)$ times

$$\begin{aligned} & \int_{s=0}^{\infty} \frac{e^{-v^2/16s} + e^{-v/2}}{\sqrt{s}(s+1)} \left(1_{\{s \in [\eta v^2, \eta^{-1}v^2]^c\}} + e^{-C'M} 1_{\{s \in [\eta v^2, \eta^{-1}v^2]\}} \right) ds \\ & \leq \int_0^{\infty} \frac{e^{-v/2}}{\sqrt{s}(s+1)} ds + \int_0^{\eta v^2} \frac{e^{-v^2/16s}}{\sqrt{s}} ds + e^{-C'M} \int_{\eta v^2}^{\infty} s^{-3/2} ds + \int_{\eta^{-1}v^2}^{\infty} s^{-3/2} ds \\ & \leq C \left(e^{-v/2} + \frac{e^{-1/16\eta}}{v\sqrt{\eta}} + \frac{e^{-C'M}}{v\sqrt{\eta}} + \frac{\sqrt{\eta}}{v} \right). \end{aligned} \quad (5.42)$$

Therefore the second integral is bounded above by $Ct^{-1}e^{\sqrt{2}v}(e^{-C'M}\eta^{-1/2} + \sqrt{\eta})$, which after a division by $t^{-1}e^{\sqrt{2}v}$ tends to 0 when $M \rightarrow \infty$ followed by $\eta \rightarrow 0$.

At the same time, thanks to the uniform convergence in Lemma 5.6 we know that as $t \rightarrow \infty$ followed by $v \rightarrow \infty$ and then $M \rightarrow \infty$, the first integral in (5.41) is asymptotic equivalent to

$$Ct^{-1}ve^{\sqrt{2}v} \int_{s=\eta v^2}^{\eta^{-1}v^2} s^{-3/2} e^{-v^2/2s} ds = Ct^{-1}e^{\sqrt{2}v} \int_{y=\eta}^{\eta^{-1}} y^{-3/2} e^{-1/2y} dy, \quad (5.43)$$

where we have substituted $y = v^2s$ to obtain the second integral. Taking now $\eta \rightarrow 0$, the last integral converges to a constant which is positive and finite.

Combining the estimate on the first integral with the bound on the second shows that the numerator is asymptotically equivalent to $Ct^{-1}e^{\sqrt{2}v}$ as $t \rightarrow \infty$ followed by $v \rightarrow \infty$. Together with the Ct^{-1} asymptotics for the denominator, this yields the desired result. \square

Lastly, we provide:

Proof of Lemma 5.3. As in the proof of Lemma 5.2 we can write the left hand side of (5.6) as the ratio between $\widetilde{\mathbb{E}}((\mathcal{C}_{t,r_t}^*([-v, 0]))^2; \widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0)$ and a quantity which is asymptotic to Ct^{-1} . Hence, it is enough to show that the last expectation is bounded above by $Ct^{-1}(v+1)e^{2\sqrt{2}v}$ for all t large enough. Again, we can use Lemma 5.4 and Lemma 5.5 to bound this expectation for fixed v and t large enough by $Ct^{-1}e^{2\sqrt{2}v}(v+1)^2$ times

$$\int_{s=0}^{\infty} \frac{e^{-v^2/16s} + e^{-v/4}}{\sqrt{s}(s+1)} \times \left(1 + \int_{s'=s}^{\infty} \frac{1}{\sqrt{s'-s}(s'-s+1)} ds' \right) ds. \quad (5.44)$$

The second integral is bounded by a constant uniformly in s . The first is bounded by a constant if $v \in [0, 1]$ and otherwise, using the substitution $s = v^2y$, by

$$Ce^{-v/4} + v^{-1} \int_{y=0}^{\infty} y^{-3/2} e^{-1/16y} dy \leq C(v+1)^{-1}. \quad (5.45)$$

All together the expectation in question is bounded above by $Ct^{-1}e^{2\sqrt{2}v}(v+1)$ whenever t is large, as we set out to prove. \square

5.2 Proof of Proposition 1.7

The proof of Proposition 1.7 will be based on the following lemma.

Lemma 5.7. *The following quantity*

$$\limsup_{t \rightarrow \infty} e^{-\sqrt{2}v} \widetilde{\mathbb{E}}_t \left(\mathcal{C}_{t,r_t}^*([-v, 0]); \max_{s \in [\eta v^2, \eta^{-1}v^2]} (h_{t-s}(X_{t-s}) - m_t + m_s) \leq -M \mid \widehat{h}_t^* = \widehat{h}_t(X_t) = 0 \right) \quad (5.46)$$

tends to 0 when $M \rightarrow \infty$ followed by $\eta \rightarrow 0$, uniformly in $v \geq 0$.

Proof. As in the proof of Lemma 5.4, we can use Lemma 3.1 and Lemma 3.2 to write the product of the above expectation with $\widetilde{\mathbb{P}}(\widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0)$ as

$$\begin{aligned} \widehat{\mathbb{E}}_t \left(\int_{s=0}^{r_t} J_{t,v}(s) \mathcal{N}(ds) ; \max_{s \in [\eta v^2, \eta^{-1}v^2]} \widehat{W}_{t,s} \leq -M \right) \\ \leq \widehat{\mathbb{E}}_t \int_{s=0}^{r_t} \left(J_{t,v}(s) 1_{[\eta v^2, \eta^{-1}v^2]^c}(s) + J_{t,v}^{\geq M}(s) 1_{[\eta v^2, \eta^{-1}v^2]}(s) \right) \mathcal{N}(ds), \end{aligned} \quad (5.47)$$

where $J_{t,v}^{\geq M}(s)$ is as in (5.17).

Using the Palm-Campbell Theorem, exactly as in Lemma 5.4, the latter is equal to

$$2 \int_{s=0}^{r_t} \left(j_{t,v}(s) 1_{[\eta v^2, \eta^{-1}v^2]^c}(s) + j_{t,v}^{\geq M}(s) 1_{[\eta v^2, \eta^{-1}v^2]}(s) \right) ds. \quad (5.48)$$

But then, as we have shown in the proof of Lemma 5.2 (display (5.42)), the latter is bounded above by $Ct^{-1}e^{\sqrt{2}v}(e^{-C'M}\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})$.

Since $\widetilde{\mathbb{P}}(\widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0) \sim Ct^{-1}$ as $t \rightarrow \infty$ by an application of Lemma 3.1 with $r = u = w = 0$ followed by the third part of Lemma 3.4, the $(t \rightarrow \infty)$ limit superior in (5.46) is bounded above by $C(e^{-C'M}\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})$ uniformly in v . \square

Proof of Proposition 1.7. In analog to (5.8), let us abbreviate $\widehat{\mathbb{P}}_t(\cdot) := \widetilde{\mathbb{P}}(\cdot \mid \widehat{h}_t(X_t) = 0)$ and $\mathcal{A}_t := \{\widehat{h}_t^* \leq 0\}$. Then, given $t \geq 0$, $v \geq 0$ and $M, \eta > 0$, we define the event

$$\mathcal{B} := \left\{ \max_{s \in [\eta v^2, \eta^{-1}v^2]} (h_{t-s}(X_{t-s}) - m_t + m_s) > -M \right\}. \quad (5.49)$$

Thanks to Lemma 5.7 for any $\epsilon > 0$, there exists $M, \eta > 0$ such that for all $v \geq 0$ if t is large enough then

$$\widehat{\mathbb{E}}_t(\mathcal{C}_{t,r_t}^*([-v, 0]); \mathcal{B}^c \mid \mathcal{A}_t) \leq \frac{\epsilon}{2} e^{\sqrt{2}v}. \quad (5.50)$$

At the same time, by Lemma 3.2 (ignoring the distribution of $\mathcal{C}_{t,r}(X_t)$) and then Lemma 3.5, we may find $\kappa' > 0$, depending on M and η , such that for all $v \geq 0$ and then t large enough,

$$\widehat{\mathbb{P}}_t(\mathcal{B} | \mathcal{A}_t) = \mathbb{P}\left(\max_{s \in [\eta v^2, \eta^{-1} v^2]} \widehat{W}_{t,s} > -M \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0, \max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0\right) \leq \frac{\kappa'}{2v}.$$

(Lemma 3.5 shows the above with $v > 1/\sqrt{\eta}$ and some κ' , which can then be increased so that the above inequality holds for all $v \geq 0$.)

Defining now the event $\widetilde{\mathcal{B}} := \{\mathcal{C}_{t,r_t}^*([-v, 0]) > \epsilon v e^{\sqrt{2}v}/\kappa'\}$, we then have

$$\begin{aligned} \widehat{\mathbb{E}}_t(\mathcal{C}_{t,r_t}^*([-v, 0]); \widetilde{\mathcal{B}}^c | \mathcal{A}_t) &\leq \widehat{\mathbb{E}}_t(\mathcal{C}_{t,r_t}^*([-v, 0]); \mathcal{B}^c | \mathcal{A}_t) + \widehat{\mathbb{E}}_t(\mathcal{C}_{t,r_t}^*([-v, 0]); \mathcal{B} \cap \widetilde{\mathcal{B}}^c | \mathcal{A}_t) \\ &\leq \frac{\epsilon e^{\sqrt{2}v}}{2} + \left(\frac{\kappa'}{2v}\right) \left(\frac{\epsilon v e^{\sqrt{2}v}}{\kappa'}\right) \leq \epsilon e^{\sqrt{2}v}, \end{aligned} \quad (5.51)$$

and also by Markov's inequality,

$$\widehat{\mathbb{P}}_t(\widetilde{\mathcal{B}} | \mathcal{A}_t) \leq \widehat{\mathbb{P}}_t(\mathcal{B} | \mathcal{A}_t) + \frac{\widehat{\mathbb{E}}_t(\mathcal{C}_{t,r_t}^*([-v, 0]); \mathcal{B}^c | \mathcal{A}_t)}{\epsilon v e^{\sqrt{2}v}/\kappa'} \leq \frac{\kappa'}{2v} + \frac{\epsilon e^{\sqrt{2}v}/2}{\epsilon v e^{\sqrt{2}v}/\kappa'} = \frac{\kappa'}{v}, \quad (5.52)$$

which is trivially smaller than $\kappa'/\epsilon v'$. The above bounds hold for all $v \geq 0$ and t sufficiently large.

Finally, use Lemma 5.1 and uniform integrability of $(\mathcal{C}_{t,r_t}^*([-v, 0]) : t \geq 0)$ under $\widehat{\mathbb{P}}_t$ as implied by Lemma 5.3 to conclude that the left hand sides of (5.51) and (5.52) converge to $\mathbb{E}(\mathcal{C}([-v, 0]); \mathcal{C}([-v, 0]) \leq \epsilon v e^{\sqrt{2}v}/\kappa')$ and $\mathbb{P}(\mathcal{C}([-v, 0]) > \epsilon v e^{\sqrt{2}v}/\kappa')$ respectively as $t \rightarrow \infty$, as long as $-v$ is not charged by \mathcal{C} with positive probability. If this is not the case, we argue as in the proof of Proposition 1.6. This shows that (1.28) and (1.29) hold with $\kappa := \kappa'/\epsilon$. \square

5.3 Proof of Proposition 1.8

Proof of Proposition 1.8. As in the proofs before, by monotonicity it is enough to show that the limit in (1.30) holds along v 's which are not charged with positive probability by \mathcal{C} . Now use Lemma 5.1 with $u = 0$, Lemma 3.1 with $r = u = w = 0$ and finally Lemma 3.2 to write for all $t \geq 0$ and such v ,

$$\mathbb{P}(\mathcal{C}([-v, 0]) = 0) = \lim_{t \rightarrow \infty} \frac{\mathbb{P}\left(\max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*} + v 1_{[0,r_t]}(\sigma_k)) \leq 0 \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0\right)}{\mathbb{P}\left(\max_{k: \sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0\right)}. \quad (5.53)$$

The denominator is asymptotic to Ct^{-1} by the third statement of Lemma 6.2 with $v = w = 0$ and $r = r_t$. It therefore remains to bound the numerator.

For a lower bound, we follow the heuristics of Brunet and Derrida and restrict the event in the numerator by intersecting with the event that up time $v/2$ there was no branching and that

at this time $\widehat{W}_{t,v/2} \leq -v$. Explicitly, we lower bound the numerator in (5.53) by

$$\begin{aligned} & \mathbb{P}(\sigma_1 > v/2, \widehat{W}_{t,v/2} \leq -v \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0) \\ & \times \mathbb{P}\left(\max_{k:\sigma_k \in [0,t]} (\widehat{W}_{t,\sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*} + v1_{[0,r_t]}(\sigma_k)) \leq 0 \mid \widehat{W}_{t,v/2} = -v, \widehat{W}_{t,t} = 0\right), \end{aligned} \quad (5.54)$$

where we have used the stochastic monotonicity of the trajectories of $\widehat{W}_{t,s}$ with respect to the initial conditions in the second term above. Now $\widehat{W}_{t,v/2}$ under $\widehat{W}_{t,0} = \widehat{W}_{t,t} = 0$ has a Gaussian law with mean $v(2t)^{-1} \log^+ t - \log^+(v/2) = -\log^+(v/2) + o(1)$ and variance $v(2t - v)/(4t) = v/2 + o(1)$, with both $o(1)$ terms tending to 0 as $t \rightarrow \infty$. At the same time σ_1 is exponential with rate 2 and independent of \widehat{W} . It follows therefore that the first probability in (5.54) will be bounded from below by

$$Ce^{-v} \frac{1}{(v/2)^{1/2}} \exp\left(-\frac{(\log^+(v/2)-v)^2}{v}\right) \geq C'v^{-\frac{1}{2}}e^{-2v}, \quad (5.55)$$

for all t large enough.

As for the second probability in (5.54), using the total probability formula with respect to W_{t,r_t} and recalling the definition of q_t from (5.21), it is at least

$$\begin{aligned} & \int_{w=-r_t^{2/3}}^{-2v} q_t((v/2, 0); (r_t, w+v)) \times q_t((r_t, w); (t, 0)) \\ & \times \mathbb{P}(\widehat{W}_{t,r_t} \in dw \mid \widehat{W}_{t,v/2} = -v, \widehat{W}_{t,t} = 0) dw. \end{aligned} \quad (5.56)$$

As we have noticed before (e.g. in the proof of Lemma 5.5), we can replace $W_{t,u}$ by $W_{r_t,u}$ in the definition of q_t , thereby obtaining,

$$q_t((v/2, 0); (r_t, w+v)) = q_{r_t}((v/2, 0); (r_t, w+v)). \quad (5.57)$$

Thanks to the asymptotic statement in Lemma 3.4, the right hand side of (5.57) is at least $Cr_t^{-1}w^-$ in the above ranges of v, w for all t large enough. The same statement also shows that $q_t((r_t, w); (t, 0)) \geq C't^{-1}w^-$ under the same conditions.

With the bounds above replacing the corresponding quantities, the last integral is equal to

$$\frac{C}{tr_t} \mathbb{E}\left(\widehat{W}_{t,r_t}^2; \widehat{W}_{t,r_t} \in [-(r_t)^{2/3}, -2v] \mid \widehat{W}_{t,v/2} = -v, \widehat{W}_{t,t} = 0\right). \quad (5.58)$$

Under the conditioning \widehat{W}_{t,r_t} is Gaussian with mean and variance given respectively by

$$(\gamma_{t,v/2} - v) \frac{t - r_t}{t - \frac{v}{2}} - \gamma_{t,r_t} = \log \frac{v}{2} - \log r_t - v + o(1) \quad \text{and} \quad (r_t - \frac{v}{2}) \frac{t - r_t}{t - \frac{v}{2}} = r_t - \frac{v}{2} + o(1), \quad (5.59)$$

with $o(1) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, for all t large enough the last expectation is at least Cr_t , making the entire expression bounded below by Ct^{-1} . Plugging this in (5.53) shows that the

numerator is at least $C't^{-1}v^{-1/2}e^{-2v}$ and in light of the asymptotics for the denominator, also that for all $v \geq 1$,

$$\mathbb{P}(\mathcal{C}([-v, 0)) = 0) \geq C'v^{-1/2}e^{-2v}. \quad (5.60)$$

We turn to an upper bound for the numerator of (5.53). Thanks to Lemma 2.9, we know that the lower tails of \widehat{h}_t^* decay uniformly in $t \geq 0$. It follows that for any $\epsilon > 0$, there must exist $M > 0$ large enough, such that $\mathbb{P}(\widehat{h}_t^* < -M) < \epsilon$. Fixing such $\epsilon > 0$ and M and assuming that $v > M$ and that $t \geq r_t^2$, we let

$$\tau = \inf \{s \geq 0 : \widehat{W}_{t,s} = -v + M\} \wedge v^2. \quad (5.61)$$

Then the numerator in (5.53) conditional on $\tau = s \leq v^2$, is at most

$$\mathbb{P}\left(\max_{k: \sigma_k \in [0, s]} \widehat{h}_{\sigma_k}^{\sigma_k^*} \leq -M\right) \mathbb{P}\left(\max_{k: \sigma_k \in [s, t]} (\widehat{W}_{t, \sigma_k} + \widehat{h}_{\sigma_k}^{\sigma_k^*}) \leq 0 \mid \widehat{W}_{t,s} = -v, \widehat{W}_{t,t} = 0\right), \quad (5.62)$$

where we have used stochastic monotonicity of W with respect to the boundary conditions and independence between W , H and \mathcal{N} .

Conditioning on $\mathcal{N}([0, s])$ and using the total probability formula, the first probability above is at most $\mathbb{E}\epsilon^{\mathcal{N}([0, s])} = e^{-2s(1-\epsilon)}$. As for the second, using the first upper bound in Lemma 3.4, we see that it is bounded above by $C(v+1)(t-s)^{-1} \leq C'(v+1)t^{-1}$, for all t large enough with C' not depending on v .

At the same time, conditional on $\widehat{W}_{t,0} = \widehat{W}_{t,t} = 0$ the distribution of $\widehat{W}_{t,s}$ is Gaussian with mean $-\gamma_{t,s} = -\log^+ s + o(1)$ and variance $s(t-s)t^{-1} = s + o(1)$ as $t \rightarrow \infty$ with both $o(1)$ tending to 0 uniformly in $s \leq v^2$. Then, setting $z := -v + M$, for all v large enough and then t large enough we have

$$\begin{aligned} \mathbb{P}(\tau \in ds \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0) / ds &\leq \mathbb{P}(\widehat{W}_{t,s} \in dz \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0) / dz \\ &\leq Cs^{-1} \exp\left(-\frac{(v-M-\log^+ s)^2}{2s}\right) \\ &\leq Cs^{-1} \exp\left(-\frac{(1-\epsilon)v^2}{2s}\right), \quad \text{whenever } s < v^2. \end{aligned} \quad (5.63)$$

Collecting the above bounds and using the total probability formula, we see that the probability of the event in the numerator of (5.53) is bounded above by

$$C \frac{(v+1)}{t} \left(\int_{s=0}^{v^2} s^{-1} e^{-(1-\epsilon)(2s + \frac{v^2}{2s})} ds + e^{-2(1-\epsilon)v^2} \right). \quad (5.64)$$

The exponent in the integrand is maximized at $s = v/2$, and its value then is $-2(1-\epsilon)v$. The last display is therefore at most $Ct^{-1}e^{-2(1-2\epsilon)v}$ for all v large enough. Together with the asymptotics for the denominator in (5.53), this shows that for any $\epsilon > 0$ if v is large enough, then

$$\mathbb{P}(\mathcal{C}([-v, 0)) = 0) \leq Ce^{-2(1-2\epsilon)v}, \quad (5.65)$$

Combining (5.60) with (5.65) shows what we wanted to prove. \square

6 Proofs of Extreme Level Set Theorems

In this section we combine the results concerning cluster properties from the previous section with the law of the limiting generalized extremal process $\widehat{\mathcal{E}}$ to derive asymptotic results for \mathcal{E} . We then use the convergence of the finite time generalized extremal process $\widehat{\mathcal{E}}_t$ to its corresponding limit, to derive asymptotic statements for the extremal level sets of h .

6.1 Structure of Extreme Level Sets

We start with a lemma that essentially contains the statement of Theorem 1.1 and Theorem 1.2. Recall the definition of $\mathcal{E}(\cdot; B)$ and $\mathcal{E}_t(\cdot; B)$ in (1.16).

Lemma 6.1. *Let C_\star be as in Proposition 1.6 and Z be as in (1.2). Then, for all $\alpha \in (0, 1]$ as $v \rightarrow \infty$,*

$$\frac{\mathcal{E}([-v, \infty); [-\alpha v, \infty) \times \mathbb{M})}{C_\star Z v e^{\sqrt{2}v}} \xrightarrow{\mathbb{P}} \alpha. \quad (6.1)$$

Proof. Given $-\infty < -v < w < z \leq \infty$, let us abbreviate

$$F_v(w, z) := \mathcal{E}([-v, \infty); [w, z] \times \mathbb{M}) = \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}} \mathcal{C}([-v - u, 0]) 1_{[w, z]}(u). \quad (6.2)$$

Since conditional on Z the law of $\widehat{\mathcal{E}}$ is that of a Poisson point process whose intensity factorizes (see (1.9)), we can write

$$\mathbb{E}(F_v(w, z) \mid Z) = \int_w^z \mathbb{E} \mathcal{C}([-v - u, 0]) Z e^{-\sqrt{2}u} du, \quad (6.3)$$

$$\mathbb{V}\text{ar}(F_v(w, z) \mid Z) = \int_w^z \mathbb{E}(\mathcal{C}([-v - u, 0]))^2 Z e^{-\sqrt{2}u} du, \quad (6.4)$$

with \mathcal{C} distributed according to ν . Using then Proposition 1.6, observing that the right hand side in the first statement of the proposition can be made into an upper bound, albeit with a different constant, we then get

$$\mathbb{E}(F_v(w, z) \mid Z) \leq C \int_w^z e^{\sqrt{2}(v+u)} Z e^{-\sqrt{2}u} du = C Z e^{\sqrt{2}v} (z - w), \quad (6.5)$$

$$\mathbb{V}\text{ar}(F_v(w, z) \mid Z) \leq C \int_w^z (v + u) e^{2\sqrt{2}(v+u)} Z e^{-\sqrt{2}u} du \leq C' Z e^{2\sqrt{2}v + \sqrt{2}z} (z + v), \quad (6.6)$$

which is valid for all v, w, z as above. Moreover,

$$\mathbb{E}(F_v(w, z) \mid Z) \sim C_\star Z e^{\sqrt{2}v} (z - w), \quad \text{as } w + v \rightarrow \infty \text{ and uniformly in } z. \quad (6.7)$$

Now given α as in the conditions of the Proposition and $v \geq 1$, let us set $w = -\alpha v$, $u = -\alpha v + \sqrt{\log v}$ and $z = \sqrt{\log v}$ and write

$$F_v(w, \infty) = F_v(w, u) + F_v(u, z) + F_v(z, \infty). \quad (6.8)$$

For the first term, we obtain from (6.5) that

$$\frac{\mathbb{E}(F_v(w, u) \mid Z)}{Zve^{\sqrt{2}v}} \leq C \frac{\sqrt{\log v}}{v} \xrightarrow{v \rightarrow \infty} 0, \quad \text{for all } v \geq 1, \quad (6.9)$$

which implies by Markov's inequality that $F_v(w, u)/(Zve^{\sqrt{2}v})$ converges to 0 as $v \rightarrow \infty$ in $\mathbb{P}(\cdot \mid Z)$ -probability for almost every Z and hence that $F_v(w, u)/(Zve^{\sqrt{2}v})$ converge to 0 in \mathbb{P} -probability. For the second term in (6.8), we use respectively (6.7) and (6.6) to obtain

$$\begin{aligned} \frac{\mathbb{E}(F_v(u, z) \mid Z)}{C_* Zve^{\sqrt{2}v}} &\sim \frac{z - u}{v} \sim \alpha \quad \text{as } v \rightarrow \infty; \\ \text{and} \quad \frac{\text{Var}(F_v(u, z) \mid Z)}{\mathbb{E}(F_v(u, z) \mid Z)^2} &\leq \frac{Ce^{\sqrt{2}z}}{Z} \frac{(z + v)}{(z - u)^2} \xrightarrow{v \rightarrow \infty} 0. \end{aligned} \quad (6.10)$$

Chebyshev's inequality then shows that $F_v(u, z)/(C_* Zve^{\sqrt{2}v}) \rightarrow \alpha$ as $v \rightarrow \infty$ in $\mathbb{P}(\cdot \mid Z)$ -probability for almost every Z and hence that $F_v(u, z)/(C_* Zve^{\sqrt{2}v}) \rightarrow \alpha$ in \mathbb{P} -probability.

Lastly, for the third term in (6.8), observe that whenever $\widehat{\mathcal{E}}([z, \infty) \times \mathbb{M}) = 0$, we also have $F_v(z, \infty) = 0$. Since conditional on Z , the intensity measure governing the law of $\widehat{\mathcal{E}}$ is finite on $[0, \infty) \times \mathbb{M}$ almost surely, the latter must happen for large enough z . This shows that $F_v(z, \infty) \xrightarrow{v \rightarrow \infty} 0$ almost surely and in particular that

$$F_v(z, \infty)(Zve^{\sqrt{2}v})^{-1} \rightarrow 0 \quad \text{as } v \rightarrow \infty \text{ in } \mathbb{P}\text{-probability.} \quad (6.11)$$

Combining the convergence results for the three terms in the left hand side of (6.8) shows that $F_v(w, \infty)(C_* Zve^{\sqrt{2}v})^{-1}$ converges in \mathbb{P} -probability to α as $v \rightarrow \infty$. Since $F_v(w, \infty)$ is precisely the left hand side of (1.17), the proof is complete. \square

The proof of Theorem 1.1 and Theorem 1.2 are now straightforward.

Proof of Theorem 1.1. The first part of the Theorem is exactly Lemma 6.1 with $\alpha = 1$, keeping in mind (1.10). For the second part, observe that the joint convergence of $(\widehat{\mathcal{E}}_t, Z_t)$ to $(\widehat{\mathcal{E}}, Z)$ together with the convergence almost surely of Z_t to Z , shows that $(\widehat{\mathcal{E}}_t, Z)$ also converges jointly weakly to $(\widehat{\mathcal{E}}, Z)$. Moreover, for any $v \geq 0$ and a Borel set $B \subset \mathbb{R} \times \mathbb{M}$, the map $\widehat{\mathcal{E}} \mapsto \mathcal{E}([-v, \infty); B)$ is continuous in the underlying topology for almost every $\widehat{\mathcal{E}}$. This is because $\widehat{\mathcal{E}}$ has a conditional Poissonian law with a product intensity measure, of which the first coordinate is absolutely continuous with respect to Lebesgue.

Since Z is almost surely positive, the latter implies that for all $v \geq 0$,

$$\frac{\widehat{\mathcal{E}}_t([-v, \infty); [-v, \infty] \times \mathbb{M})}{C_* Zve^{\sqrt{2}v}} \xrightarrow{t \rightarrow \infty} \frac{\widehat{\mathcal{E}}([-v, \infty); [-v, \infty] \times \mathbb{M})}{C_* Zve^{\sqrt{2}v}}. \quad (6.12)$$

The numerator on the right hand side is exactly $\mathcal{E}([-v, \infty))$ in light of (1.10). For the left hand side, the asymptotic separation of extreme values as manifested in (1.12) shows that we can

replace the numerator with $\mathcal{E}_t([-v, \infty))$ with the convergence still holding. This together with the first statement of the theorem yields the desired result. \square

Proof of Theorem 1.2. The first part is again an immediate consequence of Lemma 6.1. Just divide both numerator and denominator by $C_* Z v e^{\sqrt{2}v}$ for $v \geq 1$, recalling that Z is almost surely positive. Then take $v \rightarrow \infty$ and use Lemma 6.1 with the given α for the numerator and $\alpha = 1$ for the denominator. Using also relation (1.8), this gives (1.17).

As for the second part, the same argument as in the previous proof shows that the numerator and denominator in (1.18) converge weakly jointly to the numerator and denominator of (1.17) respectively. This together with the first part shows the second part of the theorem. \square

Next we turn to the proof of Theorem 1.4. Again, we start with a preparatory lemma. Recall the definition of $T_{\alpha, \kappa}(v)$ from (1.20).

Lemma 6.2. *For any $\epsilon > 0$, there exist $\kappa > 0$, such that for all $\alpha \in (0, 1)$,*

$$\lim_{v \rightarrow \infty} \mathbb{P} \left(\frac{\widehat{\mathcal{E}}(T_{\alpha, \kappa}(v))}{Z e^{\sqrt{2}\alpha v} / \sqrt{2}} < \frac{\kappa}{(1 - \alpha)2v} \right) = 1, \quad (6.13)$$

and

$$\frac{\mathcal{E}([-v, \infty); T_{\alpha, \kappa}^c(v))}{C_* Z \alpha v e^{\sqrt{2}v}} \leq \epsilon. \quad (6.14)$$

Proof. Fix $\alpha \in (0, 1)$, $v \geq 1$ and let $\epsilon' \in (0, 1/3)$. By definition of $\widehat{\mathcal{E}}$, conditional on Z , $\widehat{\mathcal{E}}(T_{\alpha, \kappa}(v))$ has a Poisson distribution with parameter given by

$$\lambda_{\alpha, \kappa}(v) = \int_{u=-\alpha v}^{\infty} \mathbb{P} \left(\mathcal{C}([-v - u, 0]) > \frac{(v+u)e^{\sqrt{2}(v+u)}}{\kappa} \right) Z e^{-\sqrt{2}u} du, \quad (6.15)$$

In the proof of proposition Proposition 1.7, precisely in display (5.52), we actually show that for every ϵ and $\kappa' > 0$ the inequality $\mathbb{P}(\mathcal{C}([-v - u, 0]) > \epsilon(v + u)e^{\sqrt{2}(v+u)}/\kappa') \leq \kappa'/(v + u)$ holds for every $u + v \geq 0$. Therefore choosing $\epsilon = 1/3$ and $\kappa' = \kappa/3$ we get

$$\lambda_{\alpha, \kappa}(v) \leq \int_{u=-\alpha v}^{\infty} \frac{\kappa}{3(v + u)} Z e^{-\sqrt{2}u} du \leq \frac{\kappa}{3\sqrt{2}(1 - \alpha)} v^{-1} Z e^{\sqrt{2}\alpha v}, \quad (6.16)$$

which is smaller than $\frac{\kappa}{2\sqrt{2}(1 - \alpha)} v^{-1} Z e^{\sqrt{2}\alpha v}$. Hence, by a second order Chebyshev's inequality

$$\mathbb{P} \left(\widehat{\mathcal{E}}(T_{\alpha, \kappa}(v)) > \frac{\kappa}{2\sqrt{2}(1 - \alpha)} v^{-1} Z e^{\sqrt{2}\alpha v} \middle| Z \right) \leq \frac{\lambda_{\alpha, \kappa}(v)}{\left(\frac{\kappa}{2\sqrt{2}(1 - \alpha)} v^{-1} Z e^{\sqrt{2}\alpha v} - \lambda_{\alpha, \kappa}(v) \right)^2} \xrightarrow{v \rightarrow \infty} 0, \quad (6.17)$$

for \mathbb{P} -almost every Z . Then using the bounded convergence theorem shows that the latter limit holds also for the unconditional measure \mathbb{P} . This conclude the proof for the first part of the lemma.

For the second, we recall $F_v(w, z)$ from (6.2) and define analogously for $-\infty < -v < w < z \leq \infty$ and $\kappa > 0$,

$$G_{v,\kappa}(w, z) := \sum_{(u, \mathcal{C}) \in \widehat{\mathcal{E}}} \mathcal{C}([-v - u, 0]) 1_{[w, z]}(u) 1_{\{\mathcal{C}([-v - u, 0]) \leq (v + u)e^{\sqrt{2}(v + u)}/\kappa\}}. \quad (6.18)$$

Clearly $G_{v,\kappa}(w, z) \leq F_v(w, z)$. Moreover, setting $z := \sqrt{\log v}$ and $w := -\alpha v$ as in Lemma 6.1, we can bound the numerator in (6.14) by $G_{v,\kappa}(w, z) + F_v(z, \infty)$.

Now, the second term was already handled in the proof of Lemma 6.1, where we have shown (6.11). As for the first, from the second part of the Proposition 1.7, still with $\epsilon'/2$, we have

$$\begin{aligned} \mathbb{E}(G_{v,\kappa}(w, z) \mid Z) &= \int_{u=w}^z \mathbb{E}\left(\mathcal{C}([-v - u, 0]); \mathcal{C}([-v - u, 0]) \leq (v + u)e^{\sqrt{2}(v + u)}/\kappa\right) Z e^{-\sqrt{2}u} du \\ &\leq \int_{u=w}^z \epsilon' e^{\sqrt{2}(v + u)} Z e^{-\sqrt{2}u} du \leq 2\epsilon' Z \alpha v e^{\sqrt{2}v}. \end{aligned}$$

In addition, as in (6.6),

$$\begin{aligned} \text{Var}(G_{v,\kappa}(w, z) \mid Z) &= \int_{u=w}^z \mathbb{E}\left(\mathcal{C}([-v - u, 0]); \mathcal{C}([-v - u, 0]) > (v + u)e^{\sqrt{2}(v + u)}/\kappa\right)^2 Z e^{-\sqrt{2}u} du \\ &\leq \text{Var}(F_v(w, z) \mid Z) \leq C Z v e^{2\sqrt{2}v + \sqrt{2}z}, \end{aligned}$$

It follows that $\text{Var}(G_v(w, z) \mid Z) / (\mathbb{E}(G_v(w, z) \mid Z))^2 \rightarrow 0$ as $v \rightarrow \infty$ and hence by Chebyshev's inequality that

$$\mathbb{P}(G_v(w, z) > 3\epsilon' Z \alpha v e^{\sqrt{2}v} \mid Z) \xrightarrow{v \rightarrow \infty} 0, \quad (6.19)$$

almost surely. By the bounded convergence theorem, the above limit holds also for the unconditional probability. Together with (6.11) this shows that as $v \rightarrow \infty$,

$$\mathbb{P}(G_v(w, z) + F_v(z, \infty) > 4\epsilon' Z \alpha v e^{\sqrt{2}v}) \rightarrow 0. \quad (6.20)$$

Since the left hand side dominates the numerator in (6.14) we just set ϵ' to be $C_\star \epsilon / 4$ in the first place and complete the proof. \square

We now prove Theorem 1.4.

Proof of Theorem 1.4. The proof follows easily from the previous lemma. Starting with the case of the limiting extremal process, given $\epsilon > 0$, we use Lemma 6.2 to find $\kappa > 0$ such that for any $\alpha \in (0, 1)$ both (6.13) and (6.14) with $\epsilon/2$ in place of ϵ hold with probability tending to 1 as $v \rightarrow \infty$. We then use (1.15) and Lemma 6.1 to claim that also

$$\frac{\mathcal{E}^*([- \alpha v, \infty))}{Z e^{\sqrt{2}\alpha v} / \sqrt{2}} > \frac{3}{4} \quad \text{and} \quad \frac{\mathcal{E}([-v, \infty); [- \alpha v, \infty) \times \mathbb{M})}{C_\star Z \alpha v e^{\sqrt{2}v}} > \frac{3}{4} \quad (6.21)$$

will hold with probability tending to 1 as $v \rightarrow \infty$.

We now observe that $\mathcal{E}^*([- \alpha v, \infty); T_{\alpha, \kappa}(v)) = \widehat{\mathcal{E}}(T_{\alpha, \kappa}(v))$ which follows by definition, and divide both the numerator and denominator of (1.22) by $Ze^{\sqrt{2}\alpha v}/\sqrt{2}$. We then see that whenever (6.13) and the first statement in (6.21) hold, we must also have (1.22). Similarly, we divide both the numerator and denominator of (1.23) by $C_* Z \alpha v e^{\sqrt{2}v}$ and observe that whenever (6.14) holds with $\epsilon/2$ and the second statement in (6.21) hold, we must also have (1.23). This completes the proof of the first part of the theorem.

Turning to the proof of the second part of the theorem, namely when the limiting objects $\mathcal{E}(\cdot; \cdot)$ and $\mathcal{E}^*(\cdot; \cdot)$ are replaced by their finite time counterparts $\mathcal{E}_t(\cdot; \cdot)$ and $\mathcal{E}_t^*(\cdot; \cdot)$ respectively. As in the previous two proofs, it is sufficient to argue that all random quantities on the left hand sides of (1.22) and (1.23) are the joint weak limits of their respective finite time analogs as $t \rightarrow \infty$. This in turn follows from similar arguments as in the previous two proofs. We omit further details. \square

Proof of Corollary 1.3. Observe that by definition for any $v > 0$ and $\alpha \in (0, 1)$, we have

$$\mathbb{P}(h_t(Y) - (m_t - v) > (1 - \alpha)v \mid \mathcal{E}_t([-v, \infty))) = \frac{\mathcal{E}_t([-v, \infty); [-\alpha v, \infty) \times \mathbb{M})}{\mathcal{E}_t([-v, \infty))}. \quad (6.22)$$

Using Theorem 1.2 the right hand side of (6.22) converges in probability to α as first $t \rightarrow \infty$ and then $v \rightarrow \infty$. Taking expectation and using the bounded convergence theorem, completes the proof. \square

6.2 Distance to the Second Maximum

Finally, let us prove the theorem concerning the distance to the second maximum.

Proof of Theorem 1.5. Starting with the first statement and assuming that \mathcal{E} is realized as in (1.5), we have

$$\{v^1 - v^2 > w\} = \{u^1 - u^2 > w\} \cap \{\mathcal{C}([-w, 0)) = 0\}. \quad (6.23)$$

Since the cluster decorations are independent of the “backbone” Poisson point process \mathcal{E}^* , the two events on the right hand side are independent and hence

$$\mathbb{P}(v^1 - v^2 > w) = \mathbb{P}(u^1 - u^2 > w) \mathbb{P}(\mathcal{C}([-w, 0)) = 0). \quad (6.24)$$

Now, to compute the first probability on the right hand side, notice that we can rewrite the intensity measure governing the law of \mathcal{E}^* as $e^{-\sqrt{2}(u - (\log Z)/\sqrt{2})} du$. This recasts \mathcal{E}^* as a randomly shifted Poisson point process with intensity measure $e^{-\sqrt{2}u} du$. This random shift is irrelevant for the quantity $u^1 - u^2$ and hence we may even assume that $Z = 1$.

In this case, by conditioning on u^1 we can write the probability that $u^1 - u^2 > w$ as

$$\int_{u=-\infty}^{\infty} e^{-\sqrt{2}u - \frac{1}{\sqrt{2}}e^{-\sqrt{2}u}} \exp\left(-\frac{e^{-\sqrt{2}u}(e^{-\sqrt{2}w} - 1)}{\sqrt{2}}\right) du = e^{-\sqrt{2}u} \int_{z=-\infty}^{\infty} e^{-\sqrt{2}z - \frac{1}{\sqrt{2}}e^{-\sqrt{2}z}} dz, \quad (6.25)$$

where we have used the substitution $z = u - w$. The integral converges to a finite positive constant showing that $\mathbb{P}(u^1 - u^2 > w) = Ce^{-\sqrt{2}w}$.

Therefore, taking the logarithm of both sides in (6.24), dividing by w and letting $w \rightarrow \infty$, the first term converges to $-\sqrt{2}$ in light of what we have just proved, while the second converges to -2 in light of Proposition 1.8. The two together show the first part of the theorem.

For the second part of the theorem, first in light of the tightness of the maximum the joint distribution of the first and second highest points of \mathcal{E}_t converge weakly to the distribution of v^1 and v^2 . It follows then by the continuous mapping theorem that the distribution of $h_t^* - h_t^{*(2)}$ converges weakly to the distribution of $v^1 - v^2$. This shows (1.25) when $w \rightarrow \infty$ along continuity points of the distribution of $v^1 - v^2$. The extension to any w follows by monotonicity following arguments similar to the ones used in the proofs before. \square

7 Proofs of Random Walk Estimates

In this section we prove the random walk estimates from Subsection 2.1, to which we refer the reader for the notation used throughout here. The starting point for all proofs is the following upper and lower bounds for the probability that a standard Brownian motion stays below a logarithmic curve. These can be found either in [12] or in more generality in [5].

Proposition 7.1. *For any $\delta > 0$ there exist $C > 0$ such that for all $t \geq 0$ and $x, y \leq 0$,*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{u \in [0,t]} (W_u - \delta^{-1} \log^+ u) \leq 0 \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t}. \quad (7.1)$$

and

$$\mathbb{P}_{0,x} \left(\max_{u \in [0,t]} (W_u - \delta^{-1} \log^+ u) \leq 0 \right) \leq C \frac{(x^- + 1)}{\sqrt{t}}. \quad (7.2)$$

Proposition 7.2. *For any $\delta > 0$ there exist $C > 0$ such that for all $t \geq 0$ and $x \leq -1$,*

$$\mathbb{P}_{0,x} \left(\max_{u \in [0,t]} (W_u + \delta^{-1} \log^+ u) \leq 0 \right) \geq \frac{C}{\sqrt{t}}. \quad (7.3)$$

7.1 The Upper Bound

The proofs follows from the sequence of lemmas given below. For the first of which we let $t, \delta > 0$, $M > 1$ and suppose that $\underline{s} = (s_k)_{k=0}^n$ for some $n \geq 1$ is a sequence of real numbers, such that with $\Delta_k := s_k - s_{k-1}$ it satisfies:

- (A1) $0 = s_0 < s_1 < \dots < s_n = t$,
- (A2) $\Delta_k \leq \delta^{-1} \log k + M$, for all $k = 1, \dots, n$.
- (A3) $\delta k - M \leq s_k \leq \delta^{-1} k + M$, for all $k = 0, \dots, n$.

We denote by $\mathcal{S}(\delta, t, M)$ the collection of all such sequences with arbitrary length n . Given such sequence $\underline{s} = (s_k)_{k=0}^n$, we define $\tilde{\underline{s}} = (\tilde{s}_k)^n$, by setting $\tilde{s}_k := t - s_k$ and also $\tilde{\Delta}_k := \tilde{s}_k - \tilde{s}_{k-1}$. Finally, we let $\tilde{\mathcal{S}}(\delta, t, M)$ be the collection of all sequence \underline{s} such that both \underline{s} and $\tilde{\underline{s}}$ are in $\mathcal{S}(\delta, t, M)$.

Lemma 7.3. *Let $\delta, t > 0$, $M > 1$ and suppose that $\underline{s} = (s_k)_{k=0}^n$ is a sequence in $\tilde{\mathcal{S}}(\delta, t, M)$. Then there exists $C = C(\delta)$ such that for all $x, y \leq 0$*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k \in [1, n-1]} \left(W_{s_k} - \delta^{-1} \log^+ \wedge^t(s_k) \right) \leq 0 \right) \leq CM^2 \frac{(x^- + 1)(y^- + 1)}{t}. \quad (7.4)$$

Proof. Noting that all constants in this proof depend only on δ , we set $\delta'^{-1} := \delta^{-1} + 2\sqrt{\delta^{-1}}$ and $M' := C_0(M + 1)$ where $C_0 > 0$ is a constant to be determined later. Using stochastic monotonicity of the trajectories of W under $\mathbb{P}_{0,x}^{t,y}$ in x, y , the left hand side in (7.4) is bounded above by

$$\frac{\mathbb{P}_{0,x}^{t,y} \left(\sup_{[0,t]} (W_u - \delta'^{-1} \log^+ \wedge^t(u)) \leq M' \right)}{1 - \sum_{k=1}^n \mathbb{P} \left(\sup_{[s_{k-1}, s_k]} W_u > M' + \frac{1}{\delta'} \log^+ (s_{k-1} \wedge (t - s_k)) \mid W_{s_{k-1}} = W_{s_k} = \frac{1}{\delta} \log^+ (s_k \wedge (t - s_{k-1})) \right)}.$$

The sum in the denominator can be further dominated by

$$\begin{aligned} & \sum_{k=1}^{n-1} \mathbb{P}_{0,0}^{\Delta_k, 0} \left(\sup_{[0, \Delta_k]} W_u > \delta'^{-1} \log^+ s_{k-1} - \delta^{-1} \log^+ s_k + M' \right) \\ & + \sum_{k=1}^{n-1} \mathbb{P}_{0,0}^{\tilde{\Delta}_k, 0} \left(\sup_{[0, \tilde{\Delta}_k]} W_u > \delta'^{-1} \log^+ \tilde{s}_{k-1} - \delta^{-1} \log^+ \tilde{s}_k + M' \right). \end{aligned} \quad (7.5)$$

By conditioning on the first time W reaches a point z and using the reflection principle, one has $\mathbb{P}_{0,0}^{s,0}(\sup_{[0,s]} W_u \geq z) = e^{-2z^2/s}$ for all $s > 0$ and $z \geq 0$. Also from assumption (A3) we know that for any $C_1 > 0$ we may choose C_0 large enough in the definition of M' , so that

$$\begin{aligned} \delta'^{-1} \log^+ s_{k-1} - \delta^{-1} \log^+ s_k + M' & \geq \delta'^{-1} \log^+ (\delta(k-1) - M) - \delta^{-1} \log^+ (\delta^{-1}k + M) + M' \\ & \geq 2\delta^{1/2}(\delta^{-1} \log k + M) + C_1, \end{aligned} \quad (7.6)$$

Using also (A2), the first sum in (7.5) is therefore bounded above by

$$C \sum_{k=1}^n \exp(-2 \log k - 2M\delta - 2C_1\delta^{1/2}) < \frac{1}{4}, \quad (7.7)$$

provided we make sure that C_1 is large enough.

Since $\tilde{\underline{s}}$ satisfies (A1)-(A3) as well, similar reasoning shows that the second sum in (7.5) can also be bounded by 1/4. But then the first display in the proof together with Proposition 7.1 shows that the probability in question is at most

$$C \frac{((x - M')^- + 1)((y - M')^- + 1)}{t} \leq CM^2 \frac{(x^- + 1)(y^- + 1)}{t}, \quad (7.8)$$

as desired. \square

Next we add the collection Y .

Lemma 7.4. *Let $\delta > 0$, $M > 1$ and $t > 0$. Suppose that $\underline{s} = (s_k)_{k=0}^n$ is a sequence in $\tilde{\mathcal{S}}(\delta, t, M)$, that W, Y are defined as in Subsection 2.1 and in particular that (2.1) holds. Then there exists $C = C(\delta)$ such that for all $x, y \leq 0$*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k \in [1, n-1]} (W_{s_k} - \delta^{-1} \log^+ \wedge^t(s_k) - Y_{s_k}) \leq 0 \right) \leq CM^3 \frac{(x^- + 1)(y^- + 1)}{t}. \quad (7.9)$$

Proof. Again, all constants in this proof will depend only on δ . Letting $\delta'^{-1} = 2\delta^{-2}$, the left hand side in (7.9) can be bounded above by

$$\begin{aligned} \sum_{m=1}^{\infty} \mathbb{P}_{0,x}^{t,y} \left(\max_{k \in [1, n-1]} (W_{s_k} - 3\delta^{-1} \log^+ \wedge^t(s_k)) \leq m \right) \\ \times \mathbb{P} \left(\max_{k \in [1, n-1]} (Y_{s_k} - 2\delta^{-1} \log^+ \wedge^t(s_k)) \in [m-1, m) \right). \end{aligned} \quad (7.10)$$

By the previous lemma, the first probability in (7.10) is at most

$$Ct^{-1}M^2(x-m)^-(y-m)^- \leq Ct^{-1}M^2m^2(x^- + 1)(y^- + 1). \quad (7.11)$$

As for the second, using the bound on the tail of the variables in Y and assumption (A3) for $\underline{s}, \tilde{\underline{s}}$, we may bound it from above by

$$\begin{aligned} \sum_{k=1}^{n-1} (\mathbb{P}(Y_{s_k} \geq 2\delta^{-1} \log^+ s_k + m - 1) + \mathbb{P}(Y_{\tilde{s}_k} \geq 2\delta^{-1} \log^+ \tilde{s}_k + m - 1)) \\ \leq \delta^{-1} \sum_{k=1}^{n-1} (e^{-2 \log^+ s_k - \delta(m-1)} + e^{-2 \log^+ \tilde{s}_k - \delta(m-1)}) \\ \leq Ce^{-\delta m} \sum_{k=1}^{\infty} ((\delta k - M) \wedge 1)^{-2} \leq CM e^{-\delta m}. \end{aligned} \quad (7.12)$$

Using both bounds in (7.10) we recover (7.9). \square

The following lemma is essentially the first part of Proposition 2.1.

Lemma 7.5. *Let $\delta, \lambda > 0$ be given and suppose that W, Y, \mathcal{N} are defined as in Subsection 2.1 and in particular that (2.1) and (2.2) hold. Then there exists $C = C(\lambda, \delta)$ such that for all $t \geq 0$, $x, y \in \mathbb{R}$,*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0, t]} (W_{\sigma_k} - \delta^{-1} \log^+ \wedge^t(\sigma_k) - Y_{\sigma_k}) \leq 0 \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t}, \quad (7.13)$$

Proof. Stochastic monotonicity of the trajectories of W under $\mathbb{P}_{0,x}^{t,y}$ in x, y implies that it is enough to prove the lemma for $x, y \leq 0$. Given a realization \mathcal{N} with $n-1 = \mathcal{N}([0, t])$, we set $s_0 = 0$, $s_k = \sigma_k$ for $k = 1, \dots, n-1$ and $s_n = t$. Set also $\delta' = \min\{\delta, 2\lambda/3, 1/(2\lambda)\}$. Then the right hand side of (7.13) is bounded from above by

$$\sum_{m=1}^{\infty} \mathbb{P}(\underline{s} \in \tilde{\mathcal{S}}(\delta', t, m+1) \setminus \tilde{\mathcal{S}}(\delta', t, m)) \times \mathbb{P}_{0,x}^{t,y} \left(\sup_{k \in [1, n-1]} W_{s_k} - \delta'^{-1} \log^+ \wedge^t(s_k) - Y_{s_k} \leq 0 \mid \underline{s} \in \tilde{\mathcal{S}}(\delta', t, m+1) \setminus \tilde{\mathcal{S}}(\delta', t, m) \right) \quad (7.14)$$

By the previous lemma, we know that the second probability is at most $C(m+1)^3(x^- + 1)(y^- + 1)/t$ for some $C > 0$ which depends only on δ . For the first probability in (7.14), we first observe that since \underline{s} and \tilde{s} have the same law, it can be bounded above by $2\mathbb{P}(\underline{s} \notin \mathcal{S}(\delta', t, m))$. The last probability is further bounded above by

$$\sum_{k=1}^{\infty} \left(\mathbb{P}(\sigma_k - \sigma_{k-1} > \delta'^{-1} \log k + m) + \mathbb{P}\left(\left|\sigma_k - \frac{k}{\lambda}\right| > \frac{k}{2\lambda} + m\right) \right). \quad (7.15)$$

Since $\sigma_k - \sigma_{k-1}$ is exponentially distributed with rate λ , the first term in the sum is bounded above by $e^{-\lambda m} k^{-3/2}$. Using the exponential Chebychev inequality, it is not difficult to see that we may find constants $C, C' > 0$ such that the second term is at most $Ce^{-C'(k+m)}$. The last two assertions imply that the sum and hence the first probability in (7.14) is exponentially decaying in m . Together with the bound on the second probability this shows what we wanted to prove. \square

As a last step for the derivation of the upper bound we allow positive values for x .

Lemma 7.6. *Let $\delta, \lambda > 0$ be given and suppose that W, Y, \mathcal{N} are defined as in the Subsection 2.1 and that in particular (2.1) and (2.2) hold. Then there exists $C = C(\lambda, \delta)$ such that for all $t \geq 0$ and all $x, y \in \mathbb{R}$ with $xy \leq 0$,*

$$\begin{aligned} & \mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0, t]} (W_{\sigma_k} - \delta^{-1} \log^+ \wedge^t(\sigma_k) - Y_{\sigma_k}) \leq 0 \right) \\ & \leq C \frac{(x^- + e^{-\sqrt{2\lambda}(1-\delta)x^+})(y^- + e^{-\sqrt{2\lambda}(1-\delta)y^+})}{t} e^{\frac{(y-x)^2}{2t}}. \end{aligned} \quad (7.16)$$

Proof. Without loss of generality, we may assume that $x \geq 0$ and $y \leq 0$. Since the family $(Y_u)_u$ have uniform upper tails, there exists $x_0 \geq 1$ such that for all $u \geq 0$ and $x \geq x_0$

$$\mathbb{P}(Y_u \geq x \frac{\delta}{2} - 2\delta^{-1} \log^+ x) \leq \frac{\delta}{2}. \quad (7.17)$$

Suppose first that $x \in [x_0, (\log^+ t)^2]$ and set $\tau := \inf\{s \geq 0 : W_s = x \frac{\delta}{2}\}$ and

$$\mathcal{A}_t(u_1, u_2) := \left\{ \max_{k: \sigma_k \in [u_1, u_2]} (W_{\sigma_k} - \delta^{-1} \log^+ \wedge^t(\sigma_k) - Y_{\sigma_k}) \leq 0 \right\}, \quad (7.18)$$

for $0 \leq u_1 < u_2 \leq t$. Then by conditioning on τ , we can write

$$\begin{aligned} \mathbb{P}_{0,x}(\mathcal{A}_t(0,t), W_t \in dy) &\leq \mathbb{P}_{0,x}(\tau \geq x^2; \mathcal{A}_t(0,x^2)) \sup_{x' \geq x\delta/2} \mathbb{P}_{x^2,x'}(\mathcal{A}_t(x^2,t); W_t \in dy) \\ &\quad + \int_{s=0}^{x^2} \mathbb{P}(\tau \in ds; \mathcal{A}_t(0,s)) \mathbb{P}_{s,x\delta/2}(\mathcal{A}_t(s,t); W_t \in dy). \end{aligned} \quad (7.19)$$

Now, if $\tau \geq s$ for some $s \leq x^2$ then the definition of τ yields

$$W_u - \delta^{-1} \log^+ \wedge^t(u) \geq x\frac{\delta}{2} - 2\delta^{-1} \log^+ x, \quad \text{for all } u \leq s. \quad (7.20)$$

By independence between W and Y , the first term in the integrand is bounded above by

$$\mathbb{P}(\tau \in ds) \mathbb{P}\left(\min_{k: \sigma_k \in [0,s]} Y_{\sigma_k} \geq x\frac{\delta}{2} - 2\delta^{-1} \log^+ x\right) \leq \mathbb{P}(\tau \in ds) e^{-\lambda s(1-\frac{\delta}{2})}, \quad (7.21)$$

where we have used (7.17) and the total probability formula, to bound the second probability on the left hand side by $\mathbb{E}(\delta/2)^{\mathcal{N}([0,s])}$. At the same time, the Gaussian density implies that

$$\mathbb{P}_{0,x}(\tau \in ds)/ds \leq \mathbb{P}_{0,x}((2\delta^{-1})W_s \in dx)/dx \leq C s^{-1/2} \exp\left(-\frac{x^2(1-\delta/2)^2}{2s}\right). \quad (7.22)$$

Similar reasoning shows that the first term on the right hand side of the inequality in (7.19) is bounded up to multiplicative constants by $e^{-\lambda x^2(1-\delta/2)}$.

Finally, concavity of the logarithm entails that $\log^+ \wedge^t(u) \leq 1 + \log^+ \wedge^{t-s}(u-s) + \log^+ s$ for all $0 \leq s \leq u \leq t$. Using this plus stochastic monotonicity of the trajectories of W in the boundary conditions to replace $x\delta/2$ by 0, the standard Gaussian density formula and shift invariance for both W and \mathcal{N} the last term in the integrand of (7.19) is bounded above by

$$C(t-s)^{-1/2} \mathbb{P}_{0,0}^{t-s,y}\left(\max_{k: \sigma_k \in [0,t-s]} (W_{\sigma_k} - \delta^{-1}(1 + \log^+ \wedge^{t-s}(\sigma_k) + \log^+ s) - Y_{s+\sigma_k}) \leq 0\right). \quad (7.23)$$

We can use the previous lemma and assumption $s < x^2$ to bound the above display by

$$C t^{-3/2} (\log^+ s + 1)(y^- + \log^+ s + 1) \leq C t^{-3/2} (\log^+ x + 1)^2 (y^- + 1). \quad (7.24)$$

For similar reasons, the last expression can also be used to bound the supremum in (7.19). Putting all these bounds together, the right hand side of (7.19) is bounded above by

$$C t^{-3/2} (\log^+ x + 1)^2 (y^- + 1) \left(e^{-\lambda x^2(1-\delta/2)} + \int_{s=0}^{\infty} e^{-\lambda s(1-\delta/2)} s^{-1/2} e^{-\frac{(x(1-\delta/2))^2}{2s}} ds \right). \quad (7.25)$$

The exponent in the integrand is maximized at $s = x(1 - \frac{\delta}{2})(2\lambda(1 - \frac{\delta}{2}))^{-\frac{1}{2}}$ making the integral bounded above by $C \exp(-\sqrt{2\lambda}(1-\delta/2)(1-\delta/2)x) \leq C \exp(-\sqrt{2\lambda}(1-\delta)x)$. Consequently, increasing x_0 if necessary, the left hand side of (7.19) is also bounded by

$$C t^{-3/2} (y^- + 1) e^{-\sqrt{2\lambda}(1-2\delta)x}, \quad \text{for all } x \in [x_0, (\log^+ t)^2]. \quad (7.26)$$

If $x > (\log^+ t)^2$ then noting that $\{\tau \leq t\} \subseteq \{W_t \in dy\}$, we bound the left hand side of (7.19) by

$$\int_{s=0}^t \mathbb{P}(\tau \in ds, \mathcal{A}_t(0, s)) \quad (7.27)$$

Proceeding as before, the integral is bounded above by $e^{-\sqrt{2\lambda}(1-\delta)x}$, which can be made smaller than (7.26) for all t and $x > (\log^+ t)^2$, by modifying the constant in (7.26) appropriately. Dividing this bound by $\mathbb{P}_{0,x}(W_t \in dy) = (2\pi t)^{-1/2} e^{-\frac{(y-x)^2}{2t}}$ gives (7.16) for the case $x > x_0$.

For $x < x_0$, we use stochastic monotonicity of W in the boundary conditions under the conditional measure, to replace x with 0. This can only increase the probability in question. Then we apply the previous Lemma 7.5 to bound the left hand side by (7.16) by $Ct^{-1}(y^- + 1)$. Increasing C if necessary, this can be made smaller than the right hand side of (7.16) for all $x \in [0, x_0]$, $y \leq 0$ and $t \geq 0$. The result follows. \square

The proof of Proposition 2.1 is now straightforward.

Proof of Proposition 2.1. Starting with the first upper bound, the probability on the left hand side of (2.4) can be written as

$$\mathbb{P}_{0,x'}^{t,y'} \left(\max_{k: \sigma_k \in [0,t]} (W_{\sigma_k} - \delta^{-1} \log^+ \wedge^t(\sigma_k) - Y_{\sigma_k}) \leq 0 \right). \quad (7.28)$$

with $x' := x - \delta^{-1}$ and $y' := y - \delta^{-1}$. Then, thanks to Lemma 7.5, the last display is at most

$$C \frac{(x'^- + 1)(y'^- + 1)}{t} \leq C \frac{(x^- + \delta^{-1} + 1)(y^- + \delta^{-1} + 1)}{t} \leq C\delta^{-2} \frac{(x^- + 1)(y^- + 1)}{t}, \quad (7.29)$$

which proves the first statement in the proposition.

To show (2.5), we proceed similarly, using this time Lemma 7.6 to bound the probability in (7.28) by the right hand side of (7.16) with x', y' replacing x, y . It is not difficult to see that the latter bound can be converted to the form appearing in the right hand side of (2.5) by adjusting the constant appropriately. \square

7.2 Entropic Repulsion and Asymptotics

The upper bound in Proposition 2.1 can be used to derive the following lemmas concerning the entropic repulsion effect. The first lemma is rather sharp and hence requires additional regularity conditions on the curve. This will be used directly to derive Proposition 2.5. The second lemma is much coarser and does not require any additional conditions. This lemma will be used as a key input to Proposition 2.2. As the arguments of both are similar, we shall only prove the first and harder of the two. For what follows, let us define for fixed $\epsilon > 0$ and all $t \geq 0$ the set,

$$R_\epsilon(t) := \{(x, y) : x, y \leq 1/\epsilon, (x^- + 1)(y^- + 1) \leq t^{1-\epsilon}\}, \quad (7.30)$$

corresponding to range at which uniformity may be obtained. It will be also convenient to define in analog to $\mathcal{A}_t(u_1, u_2)$ the abbreviation

$$\mathcal{Q}_t(u_1, u_2) := \left\{ \max_{k: \sigma_k \in [u_1, u_2]} (W_{\sigma_k} - \gamma_{t, \sigma_k} - Y_{\sigma_k}) \leq 0 \right\}, \quad (7.31)$$

for $0 \leq u_1 < u_2 \leq t$, that will be extensively used throughout this subsection.

Lemma 7.7. *Let δ, λ be given and suppose that W, Y, \mathcal{N} and γ are defined as in the Subsection 2.1 and that in particular (2.1), (2.2), (2.7) and (2.14) hold. Then for any $\epsilon > 0$ there exists $C = C(\lambda, \delta, \epsilon)$ such that for all $M > 0$, $2 < s \leq t/2$ and $x, y \in R_\epsilon(t)$,*

$$\mathbb{P}_{0,x}^{t,y} \left(\max_{u \in [s, t-s]} (W_u - \gamma_{t,u}) \geq -M, \mathcal{Q}_t(0, t) \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t} \frac{(M + 1)^2}{\sqrt{s}}. \quad (7.32)$$

Proof. We shall show that for all $r \in [s - 1, t - s]$,

$$\mathbb{P}_{0,x}^{t,y} \left(\mathcal{Q}_t(0, t), \max_{q \in [r, r+1]} (W_q - \gamma_{t,q}) \geq -M \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t(r \wedge (t - r))^{3/2}} (M + 1)^2. \quad (7.33)$$

Then summing both sides from $r = \lfloor s \rfloor$ to $r = \lceil t - s \rceil - 1$ and using the union bound will yield the desired statement.

To this end, for any $r \in [s - 1, t/2 - 1]$ and $z \in \mathbb{R}$, we may write

$$\mathbb{P}_{0,x}^{t,y}(\mathcal{Q}_t(0, t), W_r \in dz) = \mathbb{P}_{0,x}^{r,z}(\mathcal{Q}_t(0, r)) \times \mathbb{P}_{r,z}^{t,y}(\mathcal{Q}_t(r, t)) \times \mathbb{P}_{0,x}^{t,y}(W_r \in dz). \quad (7.34)$$

Since W_u is Gaussian, we may replace it by $W_u + \frac{u}{r} \gamma_{t,r}$ everywhere in the first probability on the right hand side (including the conditioning). Then we use the first part of Condition (2.14) to obtain

$$\mathbb{P}_{0,x}^{r,z}(\mathcal{Q}_t(0, r)) \leq \mathbb{P}_{0,x}^{r,z'}(\mathcal{A}_r(0, r)), \quad (7.35)$$

where $z' = z - \gamma_{t,r}$. Similarly, replacing W_u by $W_u + \gamma_{t,r}(t - u)/(t - r)$, using the shift invariance of W and \mathcal{N} and the second part of condition (2.14) show that

$$\mathbb{P}_{r,z}^{t,y}(\mathcal{Q}_t(r, t)) \leq \mathbb{P}_{0,z'}^{t-r,y}(\mathcal{A}_{t-r}(0, t - r)), \quad (7.36)$$

with z' as before. Now, if $z' \leq 0$ we can use the first upper bound in Proposition 2.1 to bound the product of both probabilities above by,

$$C \frac{(x^- + 1)(z'^- + 1)^2(y^- + 1)}{r(t - r)}. \quad (7.37)$$

If $z' > 0$ and $x \leq y$, we use the first upper bound in Proposition 2.1 for the right hand side of (7.35) and the second upper bound in the proposition for the right hand side of (7.36). In this case the product of the two probabilities is upper bounded by

$$C \frac{(x^- + 1)(y^- + 1)}{r(t - r)} e^{-Cz'} \exp\left(\frac{(z' + y^-)^2}{2(t - r)}\right). \quad (7.38)$$

Above, we have also used the stochastic monotonicity of W with respect to the initial conditions, to replace y by $y \wedge 0 = -y^-$ before applying the upper bound. If $z' > 0$ but $x > y$, we use the bounds in the opposite way, thereby obtaining the bound

$$C \frac{(x^- + 1)(y^- + 1)}{r(t - r)} e^{-Cz'} \exp\left(\frac{(z' + x^-)^2}{2r}\right), \quad (7.39)$$

on the product of the probabilities.

Next we bound $\mathbb{P}_{0,x}^{t,y}(W_r \in dz)$. Since W_r under $\mathbb{P}_{0,x}^{t,y}$ is Gaussian with variance $r(t - r)/t \in [\wedge^t(r)/2, \wedge^t(r)]$ and mean $t^{-1}(xr + y(t - r))$, we can bound

$$\frac{\mathbb{P}_{0,x}^{t,y}(W_r \in dz)}{dz} \leq \begin{cases} Cr^{-1/2} & \text{if } z' \leq 0, \\ Cr^{-1/2} \exp\left(-\frac{(z - (x \vee y))^2}{2\wedge^t(r)}\right) & \text{if } z' > 0. \end{cases} \quad (7.40)$$

where C may depend on ϵ . Multiplying this by either (7.37), (7.38) or (7.39) as prescribed above, we obtain

$$\mathbb{P}_{0,x}^{t,y}(\mathcal{Q}_t(0, t), W_r \in dz)/dz \leq C \frac{(x^- + 1)(y^- + 1)}{tr^{3/2}} (z'^- + 1)^2 e^{-Cz'^+}, \quad (7.41)$$

which is valid for all z' . Finally, integrating the left hand side above from $\gamma_{t,r} - M$, with $M \geq 1$ to ∞ gives

$$\mathbb{P}_{0,x}^{t,y}(W_r \geq \gamma_{t,r} - M, \mathcal{Q}_t(0, t)) \leq C \frac{(x^- + 1)(y^- + 1)}{tr^{3/2}} (M + 1)^2. \quad (7.42)$$

In a similar way, if $r \in [s - 1, t/2 - 1]$ and $z, w \in \mathbb{R}$ satisfy $z \leq \gamma_{t,r}$ and $w \leq \gamma_{t,r+1}$, we can bound above $\mathbb{P}_{0,x}^{t,y}(\mathcal{Q}_t(0, t), W_r \in dz, W_{r+1} \in dw)$ by

$$\begin{aligned} & \mathbb{P}_{0,x}^{r,z}(\mathcal{Q}_t(0, r)) \times \mathbb{P}_{r+1,w}^{t,y}(\mathcal{Q}_t(r + 1, t)) \times \mathbb{P}_{0,x}^{t,y}(W_r \in dz, W_{r+1} \in dw) \\ & \leq C \frac{(x^- + 1)(y^- + 1)(z'^- + 1)(w'^- + 1)e^{-C'(z-w)^2}}{tr^{3/2}} dz dw, \end{aligned} \quad (7.43)$$

where we let $z' = z - \gamma_{t,r}$, $w' = w - \gamma_{t,r+1}$ and bounded the probability on the left hand side by $Cr^{-1/2}e^{-C(z-w)^2}$ using the assumptions on x and y . At the same time, it follows from (2.14) that $|\gamma_{t,r} - \gamma_{t,q}| \leq C$ for all $1 \leq r \leq q \leq r + 1 \leq t - 1$ for some $C = C(\delta)$. Therefore, for r, z, w as above, the reflection principle for Brownian motion together with stochastic monotonicity imply

$$\mathbb{P}_{r,z}^{r+1,w}\left(\max_{q \in [r, r+1]} W_q - \gamma_{t,q} \geq -M\right) \leq \mathbb{P}_{0,z' \vee w'}^{1,z' \vee w'}\left(\max_{q \in [0, 1]} W_q \geq -M - C\right) \leq Ce^{-C'(-M - z' \vee w')^2}. \quad (7.44)$$

Using the last two bounds, we may write for all $r \in [s, t/2]$,

$$\begin{aligned}
& \mathbb{P}_{0,x}^{t,y} \left(\mathcal{Q}_t(0,t), \max_{q \in [r, r+1]} (W_q - \gamma_{t,q}) \geq -M, W_r - \gamma_{t,r} \leq -M, W_{r+1} - \gamma_{t,r+1} \leq -M \right) \\
& \leq \int_{z,w} \mathbb{P}_{0,x}^{t,y} (\mathcal{Q}_t(0,t), W_r \in dz, W_{r+1} \in dw) \mathbb{P}_{r,z}^{r+1,w} \left(\max_{q \in [r, r+1]} (W_q - \gamma_{t,q}) \geq -M \right) \\
& \leq C \frac{(x^- + 1)(y^- + 1)}{tr^{3/2}} \int_{z', w' \leq -M} (z'^- + 1)(w'^- + 1) e^{-C'((z' - w')^2 + (M + z' \vee w')^2)} dz' dw', \\
& \leq C \frac{(x^- + 1)(y^- + 1)}{tr^{3/2}} (M + 1)^2
\end{aligned} \tag{7.45}$$

with $z \leq \gamma_{t,r} - M$, $w \leq \gamma_{t,r+1} - M$ in the first integral and we have also used that $|z - w| > |z' - w'| - C$ which follows from the bound on the difference $|\gamma_{t,r} - \gamma_{t,r+1}|$ as shown above.

Summing the last display together with display (7.42) twice – for r and $r + 1$, and using the union bound, we get (7.33) for $r \in [s - 1, t/2 - 1]$. A symmetric argument will show the same for $r \in [t/2, t - s]$. \square

The second entropic repulsion result is provided in the following lemma.

Lemma 7.8. *Let δ, λ be given and suppose that W, Y, \mathcal{N} and γ are defined as in the Subsection 2.1 and that in particular (2.1) and (2.2) hold. Then for any $\epsilon > 0$ and $M > 0$, there exists $C = C(\lambda, \delta, \epsilon, M) > 0$ such that for all $2 < s \leq t/2$ and $x, y \in R_\epsilon(t)$,*

$$\begin{aligned}
& \mathbb{P}_{0,x}^{t,y} \left(\left\{ \max_{\sigma_k \in [0, t]} W_{\sigma_k} - \delta^{-1}(1 + \log^+ \wedge^t(\sigma_k)) - Y_{\sigma_k}^+ \leq 0 \right\} \right. \\
& \quad \left. \cap \left(\left\{ \max_{[s, t-s]} W_u \geq 0 \right\} \cup \left\{ \max_{\sigma_k \in [s, t-s]} W_{\sigma_k} + Y_{\sigma_k}^- \geq -M \right\} \right) \right) \leq C \frac{(x^- + 1)(y^- + 1)}{t} s^{-1/4}.
\end{aligned} \tag{7.46}$$

Proof. The proof uses similar arguments as in the proof of the previous lemma. We therefore omit it. \square

We now turn to the asymptotic statements of Subsection 2.1, namely Propositions 2.2 and 2.4, but first a few definitions are in order. In analog to (7.31) let us define for $0 \leq u_1 < u_2 < t$ the event,

$$\mathcal{Q}_t^-(u_1, u_2) := \left\{ \max_{\sigma_k \in [u_1, u_2]} (W_{\sigma_k} - \gamma_{t, t - \sigma_k} - Y_{t - \sigma_k}) \leq 0 \right\}. \tag{7.47}$$

We shall also need the $t = \infty$ analogs of $\mathcal{Q}_t(u_1, u_2)$ and $\mathcal{Q}_t^-(u_1, u_2)$, which we define as

$$\begin{aligned}
& \mathcal{Q}_\infty(u_1, u_2) := \left\{ \max_{\sigma_k \in [u_1, u_2]} (W_{\sigma_k} - \gamma_{\infty, \sigma_k} - Y_{\sigma_k}) \leq 0 \right\}; \\
& \text{and } \mathcal{Q}_\infty^-(u_1, u_2) := \left\{ \max_{\sigma_k \in [u_1, u_2]} (W_{\sigma_k} - \gamma_{\infty, -\sigma_k} - Y_\infty^{(\sigma_k)}) \leq 0 \right\},
\end{aligned} \tag{7.48}$$

where $0 \leq u_1 < u_2$ and $(Y_\infty^{(u)} : u \geq 0)$ is an independent collection of random variables with $Y_\infty^{(u)} \stackrel{\text{law}}{=} Y_\infty$, which we assume to be defined on the same probability space as W, \mathcal{N}, Y and independent of W and \mathcal{N} . Finally, we shall also need

$$f_s(x) := \mathbb{E}_{0,x}(W_s^-; \mathcal{Q}_\infty(0, s)) \quad , \quad g_s(y) := \mathbb{E}_{0,y}(W_s^-; \mathcal{Q}_\infty^-(0, s)) \quad , \quad (7.49)$$

for any $s \geq 0$ and $x, y \in \mathbb{R}$. The proof of Proposition 2.2 will rely on the following lemmas.

Lemma 7.9. *Let $\delta, \lambda > 0$ and suppose that W, Y, \mathcal{N} and γ are defined as in Subsection 2.1 and in particular that (2.1), (2.2), (2.6), (2.7) and (2.8) hold. Then for any $\epsilon > 0$,*

$$\lim_{s \rightarrow \infty} \limsup_{t \rightarrow \infty} \sup_{x, y \in R_\epsilon(t)} \frac{t}{(x^- + 1)(y^- + 1)} \left| \mathbb{P}_{0,x}^{t,y}(\mathcal{Q}_t(0, t)) - \frac{2f_s(x)g_s(y)}{t} \right| = 0. \quad (7.50)$$

Lemma 7.10. *Let $\delta, \lambda > 0$ and suppose that W, Y, \mathcal{N} and γ are defined as in Subsection 2.1 and in particular that (2.1), (2.2), (2.6), (2.7) and (2.8) hold. Then for all $x, y \in \mathbb{R}$,*

$$\liminf_{s \rightarrow \infty} f_s(x) > 0 \quad , \quad \liminf_{s \rightarrow \infty} g_s(y) > 0. \quad (7.51)$$

Moreover for each $s \geq 1$,

$$\lim_{x \rightarrow \infty} \frac{f_s(-x)}{x} = \lim_{y \rightarrow \infty} \frac{g_s(y)}{y} = 1. \quad (7.52)$$

Before proving these lemmas, let us use them to prove Proposition 2.2.

Proof of Proposition 2.2. Fix $\epsilon > 0$ and let us first take also fixed $x, y < 1/\epsilon$. Then using Lemma 7.9 and distributing the t factor in (7.50), the absolute value becomes the difference between two functions: $t \mapsto t\mathbb{P}_{0,x}^{t,y}(\mathcal{A}_t(0, t))$ and $s \mapsto 2f_s(x)g_s(y)$. Then the vanishing of the limit in (7.50) shows that in fact both functions are Cauchy in their respective arguments and hence converging to an identical limit:

$$\lim_{t \rightarrow \infty} t\mathbb{P}_{0,x}^{t,y}(\mathcal{A}_t(0, t)) = \lim_{s \rightarrow \infty} 2f_s(x)g_s(y) =: 2h_\infty(x, y) < \infty, \quad (7.53)$$

which is finite (but possibly zero).

Taking $Y'_u \stackrel{\text{law}}{=} Y_\infty$ and $\gamma'_{t,u} := \gamma_{\infty, -\wedge^t(u)}$ for all $0 \leq u \leq t$, Conditions (2.6), (2.7) and (2.8) still hold with $Y'_\infty \stackrel{\text{law}}{=} Y_\infty$ and $\gamma'_{\infty,u} = \gamma'_{\infty,-u} = \gamma_{\infty,-u}$. Moreover, denoting by f'_s and g'_s the corresponding analogs of f_s and g_s from (7.49), we have $f'_s(y) = g'_s(y) = g_s(y)$ for all $s \geq 0$ and $y < 1/\epsilon$. Since Lemma 7.9 is still in force with Y' and γ' in place of Y and γ , the following limit exists for all $y < 1/\epsilon$

$$\lim_{s \rightarrow \infty} f'_s(y)g'_s(y) = \lim_{s \rightarrow \infty} g_s(y)^2 \quad (7.54)$$

It follows that $g(y) := \lim_{s \rightarrow \infty} g_s(y)$ exists and is finite for all such y and in light of the first part of Lemma 7.10 also positive. Returning to our original Y and γ , this also shows that $f(x) := \lim_{s \rightarrow \infty} f_s(x)$ exists for all $x < 1/\epsilon$, that it is finite and positive (by Lemma 7.10 again) and moreover, that $h_\infty(x, y) = f(x)g(y)$.

Turning to the issue of uniformity, the fact that the limit in (7.50) is uniform with respect to $x, y \in R_\epsilon(t)$ in (7.50) shows that

$$|f_s(x)g_s(y) - f(x)g(y)| = (x^- + 1)(y^- + 1)o(1) \quad \text{as } s \rightarrow \infty, \quad (7.55)$$

uniformly in all $x, y < 1/\epsilon$ and

$$\left| \mathbb{P}_{0,x}^{t,y}(\mathcal{A}_t(0,t)) - \frac{2f(x)g(y)}{t} \right| = \frac{(x^- + 1)(y^- + 1)}{t}o(1) \quad \text{as } t \rightarrow \infty, \quad (7.56)$$

uniformly in $x, y \in R_\epsilon(t)$. Using (7.55) with $y = 0$, it follows from the uniformity of the $o(1)$ term that

$$\lim_{s \rightarrow \infty} \limsup_{x \rightarrow -\infty} \left| \frac{f_s(x)}{x^- + 1} g_s(0) - \frac{f(x)}{x^- + 1} g(0) \right| = 0. \quad (7.57)$$

Using this together with the second part of Lemma 7.10 and a similar argument with the roles of f and g reversed, yields

$$\lim_{x \rightarrow \infty} f(-x)/x = \lim_{y \rightarrow \infty} g(-y)/y = 1. \quad (7.58)$$

Now, together with the positivity of $f(x)$ and $g(y)$ the latter implies that there exists $C > 0$ such that $f(x) > C(x^- + 1)$ and $g(y) > C(y^- + 1)$ for all $x, y < 1/\epsilon$. In turn, we then have that the right hand side of (7.56) is actually $o(f(x)g(y)/t)$ uniformly in $x, y \in R_\epsilon(t)$ as $t \rightarrow \infty$. This shows (2.9) with f and g , while (7.58) is exactly (2.10). \square

It remains therefore to prove Lemma 7.9 and Lemma 7.10.

Proof of Lemma 7.9. Let $A \triangle B$ denote the symmetric difference of A and B , then given $s \geq 1$ we observe that for any $t \geq 2s$,

$$\begin{aligned} & \left\{ \max_{k: \sigma_k \in [0,t]} W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k} \leq 0 \right\} \triangle \left\{ \max_{k: \wedge^t(\sigma_k) \leq s} W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k} \leq 0; \max_{[s,t-s]} W_u \leq 0 \right\} \\ & \subset \left\{ \max_{\sigma_k \in [0,t]} W_{\sigma_k} - \delta^{-1}(1 + \log^+ \wedge^t(\sigma_k)) - Y_{\sigma_k}^+ \leq 0 \right\} \\ & \quad \cap \left(\left\{ \max_{[s,t-s]} W_u \geq 0 \right\} \cup \left\{ \max_{\sigma_k \in [s,t-s]} W_{\sigma_k} + Y_{\sigma_k}^- \geq -\delta^{-1} \right\} \right), \quad (7.59) \end{aligned}$$

where we have used the bounds on $\gamma_{t,u}$ per (2.7). But then, (the entropic repulsion) Lemma 7.8 with $M = \delta^{-1}$ yields

$$\begin{aligned} & \limsup_{\substack{t \rightarrow \infty \\ s \rightarrow \infty}} \sup_{x, y \leq \epsilon^{-1}} \frac{t}{(x^- + 1)(y^- + 1)} \left| \mathbb{P}_{0,x}^{t,y} \left(\max_{k: \sigma_k \in [0,t]} W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k} \leq 0 \right) \right. \\ & \quad \left. - \mathbb{P}_{0,x}^{t,y} \left(\max_{k: \wedge^t(\sigma_k) \leq s} W_{\sigma_k} - \gamma_{t,\sigma_k} - Y_{\sigma_k} \leq 0; \max_{[s,t-s]} W_u \leq 0 \right) \right| = 0, \quad (7.60) \end{aligned}$$

with the order in the limits taken from top to bottom. To estimate the second probability above, we claim that we can add the restriction $\{|W_s - x| \vee |W_{t-s} - y| < \log t\}$ to the corresponding

event at a uniform $o(t^{-1})$ cost. Indeed, since W_s and W_{t-s} are Gaussian under $\mathbb{P}_{0,x}^{t,y}$ with variance $s(t-s)/t \leq s$ and means $x + (y-x)s/t$ and $y + (x-y)s/t$ respectively. Since $(y-x)/t \rightarrow 0$ as $t \rightarrow \infty$, uniformly in $x, y \in R_\epsilon(t)$, it follows from the standard Gaussian tail estimate and union bound that

$$\mathbb{P}_{0,x}^{t,y}(|W_s - x| \vee |W_{t-s} - y| > \log^+ t) \leq C \exp\left(-\frac{\log^2 t}{s}\right) = o(t^{-1}) \quad (7.61)$$

as $t \rightarrow \infty$, uniformly in $x, y \in R_\epsilon(t)$ for fixed $s \geq 0$.

If $z, w \leq 0$ satisfy $|z - x| \vee |w - y| < \log t$, then $z^- w^- \leq t^{1-\epsilon/2}$ for all t large enough, so we use the reflection principle for standard Brownian motion to obtain

$$\mathbb{P}_{0,x}^{t,y}\left(\max_{[s,t-s]} W_u \leq 0 \mid W_s = z, W_{t-s} = w\right) = 1 - \exp\left(-\frac{2(z^- w^-)}{t-2s}\right) = 2\frac{z^- w^-}{t}(1 + o(1)), \quad (7.62)$$

with $o(1)$ term tending to 0 as $t \rightarrow \infty$ uniformly in $x, y \in R_\epsilon(t)$ for fixed s . In this case also,

$$\frac{\mathbb{P}_{0,x}^{t,y}(W_s \in dz, W_{t-s} \in dw)}{dzdw} = \frac{\exp\left(-\frac{(z-x)^2 - (y-w)^2}{2s}\right)}{2\pi s^2} \times \sqrt{\frac{t}{(t-s)}} \exp\left(-\frac{(w-z)^2}{2(t-s)} + \frac{(x-y)^2}{2t}\right), \quad (7.63)$$

with the term after the product sign tending to 1 as $t \rightarrow \infty$ uniformly as above.

Conditioning on $(W_s, W_{t-s}) = (z, w)$ and using the total probability formula, the second probability in (7.60), restricted by $\{|W_s - x| \vee |W_{t-s} - y| < \log t\}$ is therefore equal to

$$\frac{2}{t} \mathbb{E}_{0,x}(W_s^-; \mathcal{Q}_t(0, s), |W_s - x| < \log t) \times \mathbb{E}_{0,y}(W_{t-s}^-; \mathcal{Q}_t(0, s), |W_{t-s} - y| < \log t) (1 + o(1)), \quad (7.64)$$

where we have also used the shift and time-reversal invariance in law of both W and \mathcal{N} .

Next, we wish to get rid of the restriction on $|W_s - x|$ and $|W_{t-s} - y|$. To this end, we note that $\mathbb{E}_{0,x}(W_s^-) \leq \mathbb{E}_{0,0}(W_s^-) + x^- \leq s^{1/2} + x^-$ and thanks to the Gaussian tail also

$$\mathbb{E}_{0,x}(W_s^-; |W_s - x| > \log t) \leq x^- \mathbb{P}_{0,0}(|W_s| > \log t) + \mathbb{E}_{0,0}(|W_s|; |W_s| > \log t) = o(1), \quad (7.65)$$

the latter holding as $t \rightarrow \infty$ uniformly in x in its allowed range. Since the same clearly holds for y , we may remove the events $\{|W_s - x| < \log t\}$ and $\{|W_{t-s} - y| < \log t\}$ from the expectations in (7.64) as well as the $1 + o(1)$ factor in the end of the display, at the cost of an extra term which is $o((x^- + 1)(y^- + 1)/t)$ as $t \rightarrow \infty$.

It remains to show that both expectations converge uniformly to $f_s(x)$ and $f_s^-(y)$ respectively when $t \rightarrow \infty$. Assumptions (2.6) and (2.8) and the absolute continuity of W_{σ_k} with respect to the Lebesgue measure could be used in conjunction with the dominated convergence theorem to show that this is indeed the case for fixed x, y . However, in order to show that the convergence is uniform, we need to work slightly harder. We proceed detailing the argument for the convergence of the second expectation to $g_s(y)$. The argument for the convergence of the first to $f_s(x)$ is similar and in fact easier, since the random variables Y_{σ_k} do not depend on t .

The weak convergence $Y_u \implies Y_\infty$ as $u \rightarrow \infty$, imply via a standard coupling argument, that we can assume that collections $(Y_u : u \geq 0)$ and $(Y_\infty^{(u)} : u \geq 0)$ are defined jointly in such a way that $Y_u - Y_\infty^{(u)}$ converges to 0 in probability. Letting now $\tilde{\mathcal{Q}}_\infty^-(0, s)$ be defined as $\mathcal{Q}_\infty^-(0, s)$ in (7.48) only with $Y_\infty^{(t-\sigma_k)}$ in place of $Y_\infty^{(\sigma_k)}$ and observing that the definition of $f_s^-(y)$ does not change if we use $\tilde{\mathcal{Q}}_\infty^-(0, s)$ in place of $\mathcal{Q}_\infty^-(0, s)$, we can write

$$\begin{aligned} |\mathbb{E}_{0,y}(W_s^-; \mathcal{Q}_t^-(0, s)) - g_s(y)| &\leq \mathbb{E}_{0,y}(W_s^-; \mathcal{Q}_t^-(0, s) \triangle \tilde{\mathcal{Q}}_\infty^-(0, s)) \\ &\leq (\mathbb{E}_{0,y} W_s^2)^{1/2} \left(\mathbb{P}_{0,y}(\mathcal{Q}_t^-(0, s) \triangle \tilde{\mathcal{Q}}_\infty^-(0, s)) \right)^{1/2}, \end{aligned} \quad (7.66)$$

where we have used Cauchy-Schwarz for the last inequality.

Now the first term in the second line is equal to $(y^2 + s)^{1/2} \leq \sqrt{s}C(y^- + 1)$ for all $y \leq \epsilon^{-1}$. For the second term in the display, define for each $\sigma_k \in [0, s]$ the interval

$$I_{\sigma_k} := [(\gamma_{t,t-\sigma_k} + Y_{t-\sigma_k}) \wedge (\gamma_{\infty,-\sigma_k} + Y_\infty^{(t-\sigma_k)}), (\gamma_{t,t-\sigma_k} + Y_{t-\sigma_k}) \vee (\gamma_{\infty,-\sigma_k} + Y_\infty^{(t-\sigma_k)})]. \quad (7.67)$$

Then $\mathcal{Q}_t^-(0, s) \triangle \tilde{\mathcal{Q}}_\infty^-(0, s) \subset \{\exists \sigma_k \in [0, s] : W_{\sigma_k} \in I_{\sigma_k}\}$. Conditioning on $\mathcal{N}([0, s]) = i$ and Y the (conditional) probability of the latter event is smaller than $i\mathbb{P}(W_U \in I_U)$, with U is an independent uniform random variable in $[0, s]$. In particular, denoting by $|I_u|$ the Lebesgue measure of I_u (with I_u defined as above replacing σ_k by u) we have that

$$\begin{aligned} \mathbb{P}_{0,y}(W_U \in I_U) &\leq \frac{1}{s} \int_0^s 1 \wedge \left(|I_u| \times \sup_{x \in \mathbb{R}} \frac{e^{-\frac{(x-y)^2}{u}}}{\sqrt{2\pi u}} \right) du \\ &\leq \frac{C}{s} \int_0^s (u^{-\frac{1}{2}} \vee 1) \times (|I_u| \wedge 1) du. \end{aligned} \quad (7.68)$$

The length of I_u is at most $|\gamma_{\infty,-u} - \gamma_{t,t-u}| + |Y_\infty^{(t-u)} - Y_{t-u}|$ and hence $|I_u| \wedge 1$ may be further upper bounded by $|\gamma_{\infty,-u} - \gamma_{t,t-u}| + |Y_\infty^{(t-u)} - Y_{t-u}| \wedge 1$. Using the total expectation formula, the upper-bound from (7.68) and Fubini to exchange between the Lebesgue integral over u and the expected value with respect to Y we obtain

$$\begin{aligned} &\mathbb{P}_{0,y}(\exists \sigma_k \in \mathcal{N} \cap [0, s] : W_{\sigma_k} \in I_{\sigma_k}) \\ &\leq \frac{C\mathbb{E}[\mathcal{N}([0, s])]}{s} \int_0^s (u^{-\frac{1}{2}} \vee 1) \times \left(|\gamma_{\infty,-u} - \gamma_{t,t-u}| + \mathbb{E}[|Y_\infty^{(t-u)} - Y_{t-u}| \wedge 1] \right) du \\ &\leq C\lambda s \times \sup_{u \in [0, s]} \left(|\gamma_{\infty,-u} - \gamma_{t,t-u}| + \mathbb{E}[|Y_\infty^{(t-u)} - Y_{t-u}| \wedge 1] \right). \end{aligned} \quad (7.69)$$

In light of (2.7) and our coupling construction, the right hand side of (7.69) goes to 0 when $t \rightarrow \infty$. The above upper bounds together with (7.66) yield

$$\limsup_{t \rightarrow \infty} \frac{|\mathbb{E}_{0,y}(W_s^-; \mathcal{Q}_t^-(0, s)) - g_s(y)|}{(y^- + 1)} = 0, \quad (7.70)$$

uniformly in $x, y \in R_\epsilon(t)$. A similar argument, with the obvious changes, applies to $f_s(x)$. Since $\mathbb{E}_{0,x}(W_s^-) \leq C(x^- + 1)$ and $\mathbb{E}_{0,y}(W_s^-) \leq C(y^- + 1)$, we can further replace the product of the expectations in (7.64) by $f_s(x)g_s(y)$ at the cost of another $o((x^- + 1)(y^- + 1)/t)$ uniformly for all $x, y < 1/\epsilon$. Collecting all error terms we obtain (7.50). \square

Proof of Lemma 7.10. Beginning with (7.51), we shall prove only the first inequality, as the argument for the proof of the second is identical. Observing that $f_s(x)$ is decreasing in x , it is enough to consider positive x -s. We now fix such $x \geq 0$, let $M > 0$ and use the independence between W, Y and \mathcal{N} , the Markov property for W to bound $f_{s+1}(x)$ from below for any $s \geq 1$ by

$$\begin{aligned} \mathbb{P}(\sigma_1 \geq 2) \mathbb{P}_{0,x}(W_1 \leq -\delta^{-1} - 1) \mathbb{P}_{0,x} \left(\min_{k: \sigma_k \in [2, s+1]} (Y_{\sigma_k} + M \log(\sigma_k - 1)) \geq 0 \right) \\ \times \mathbb{E}_{1,-1} \left(W_{s+1}^-; \max_{u \in [1, s+1]} (W_u + M \log^+(u - 1)) \leq 0 \right). \end{aligned} \quad (7.71)$$

Above we have also used stochastic monotonicity of W in the initial condition.

The first two terms in (7.71) are positive for any x . To lower bound the third, we condition on \mathcal{N} and use Assumption (2.1) and independence to bound it below by

$$\mathbb{E} \exp \left(\int_{u=2}^s \log(1 - \delta^{-1} e^{-\delta M \log(u-1)}) \mathcal{N}(du) \right) = \exp \left(-\lambda \delta^{-1} \int_{u=2}^s (u-1)^{-M\delta} du \right), \quad (7.72)$$

where we have used the explicit formula for the Laplace transform of \mathcal{N} . If $M\delta > 1$, the integral up to $s = \infty$ will converge, making the above quantity uniformly bounded away from 0 for all $s \geq 2$.

It remains to show that the expectation in (7.71) is also uniformly positive. To this end, we recall (or use the second part of Proposition 7.1) that for all $z \leq -1$,

$$\mathbb{P}_{s,z} \left(\max_{u \in [s, 2s]} W_u \leq -1 \right) \leq Cs^{-1/2}(z^- + 1). \quad (7.73)$$

Then using the total probability formula with respect to W_s and using the above bound we get,

$$\mathbb{P}_{0,-1} \left(\max_{u \in [0, 2s]} (W_u + M \log^+ u) \leq 0 \right) \leq Cs^{-1/2} \mathbb{E}_{0,-1} \left(W_s^- + 1; \max_{u \in [0, s]} (W_u + M \log^+ u) \leq 0 \right) \quad (7.74)$$

By Proposition 7.2, the probability on the left is at least $Cs^{-1/2}$, while by Proposition 2.1, the expectation of 1 on the event in the expectation on the right is at most $Cs^{-1/2}$. This shows that the expectation of W_s^- on the same event is bounded away from 0 for all s large enough and by shift invariance the same is true for the expectation in (7.71). All together this shows the existence of $C > 0$ such that $f_{s+1}(x) > c$ for all s large enough, which is the first statement in the lemma.

As for (7.52), again arguing only for f , if we define $\mathcal{Q}_\infty(0, s; x)$ as in (7.48) only with $W_{\sigma_k} + x$ in place of W , then clearly we can write $f_s(-x)$ as $\mathbb{E}_{0,0}((W_s - x)^-; \mathcal{Q}_\infty(0, s; -x))$. Dividing by

x and letting $x \rightarrow \infty$, the random variable inside the expectation converges almost surely to 1 and bounded for all $x \geq 1$ by $|W_s| + 1$. The result then follows from the bounded convergence theorem. \square

Next, we prove Proposition 2.4,

Proof of Proposition 2.4. A quick look at the proof of Lemma 7.9, shows that the outer most limit in (7.50) is in fact uniform in all Y, γ satisfying (2.1) (2.6), (2.7) and (2.8) for a fixed $\delta > 0$. Indeed, the overall rate of convergence in s depends only on the rate of convergence in s in (7.60). In the rest of the argument, s remains fixed. This in turn depends on the entropic repulsion bound given in Lemma 7.8, which depends on Y and γ only through conditions (2.1) and (2.7) and as such naturally uniform for all Y, γ satisfying these conditions with a given δ . In particular, as in the proof of Proposition 2.2, this implies that $f_s(x)/(x^- + 1) \rightarrow f_\infty(x)$ and $g_s(y)/(y^- + 1) \rightarrow g_\infty(y)$ as $s \rightarrow \infty$, uniformly in $x, y < 1/\epsilon$ for fixed $\epsilon > 0$ and Y, γ for fixed $\delta > 0$.

Now, denoting by $f_s^{(r)}$ and $g_s^{(r)}$ the versions of f_s and g_s respectively defined as in (7.49) via the events in (7.48) only with $Y^{(r)}$ and $\gamma^{(r)}$ in place of Y and γ . Assumption (2.12) in the lemma, the absolute continuity of the marginals of W and the dominated convergence theorem ensure that for all $x, y \in \mathbb{R}$ and $s > 0$,

$$f_s^{(r)}(x) \xrightarrow{s \rightarrow \infty} f_s(x) \quad , \quad g_s^{(r)}(x) \xrightarrow{s \rightarrow \infty} g_s(x) , \quad (7.75)$$

with f_s, g_s defined with respect to Y and γ appearing in (2.12).

Thanks to the uniformity discussed above, we can exchange the order in which the limits are taken, thereby obtaining

$$\lim_{r \rightarrow \infty} f^{(r)}(x) = \lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} f_s^{(r)}(x) = \lim_{s \rightarrow \infty} \lim_{r \rightarrow \infty} f_s^{(r)}(x) = \lim_{s \rightarrow \infty} f_s(x) = f(x) , \quad (7.76)$$

with a similar statement for g . The last statement in the proposition is obvious in light of the definition of g . \square

Finally, we prove Proposition 2.5.

Proof of Proposition 2.5. The conditional probability is equal to the ratio between the probability in the statement of Lemma 7.7 and the probability in the statement of Proposition 2.2. In light of the upper bound in Lemma 7.7 and the limit in Proposition 2.2, noting that $f(x)$ and $g(y)$ are positive, the proof follows. \square

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